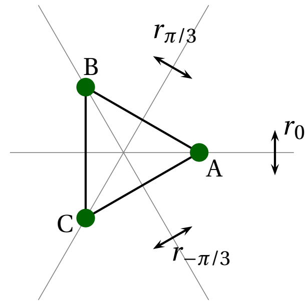
Solutions to Introductory Problems

Equilateral triangle



$$\mathsf{D}_3 = \{I, R_{2\pi/3}, R_{4\pi/3}, r_0, r_{\pi/3}, r_{-\pi/3}\}.$$

1. We know, $R_\alpha = r_{\pi/3} r_0$ and $R_\beta = r_0 r_{\pi/3}$. Therefore

$$R_\alpha R_\beta = r_{\pi/3} r_0 r_0 r_{\pi/3} r = r_{\pi/3} r_0^2 r_{\pi/3} = r_{\pi/3} r_{\pi/3} = r_{\pi/3}^2 = I.$$

2. The product (composition) $r_{\pi/3} r_0$ means first do the reflection r_0 then $r_{\pi/3}$. Consider the effect on the point A: under the reflection r_0 the vertex A remains at A (just B and C swap over). Then under $r_{\pi/3}$ A moves to B. Hence the overall effect of $r_{\pi/3} r_0$ and there is only one rotation that does that, namely $R_{2\pi/3}$. Since the questions tells us the product is a rotation, the answer is $r_{\pi/3} r_0 = R_{2\pi/3}$; that is, $\alpha = 2\pi/3$. [Note that there is no reason to use the point A, any vertex would work just as well.]

3. A similar approach shows $r_0 r_{\pi/3} = R_{-2\pi/3}$; that is, $\beta = -2\pi/3$ (or $4\pi/3$). For example, $r_{\pi/3}$ maps A to B and r_0 then maps B to C, so the overall effect is to map A to C and the only rotation that does that is $R_{-2\pi/3}$.

4. (a) The elements I and r_0 fix A.
(b) The elements I and $r_{-\pi/3}$ fix B, and the elements I and $r_{\pi/3}$ fix C.
(c) The elements $R_{2\pi/3}$ and $r_{\pi/3}$ move A to B.
(d) $\{R_{2\pi/3}, r_{\pi/3}\}$ is not a subgroup (it doesn't contain the identity element for example). The other 3 are all subgroups (of order 2).

5.

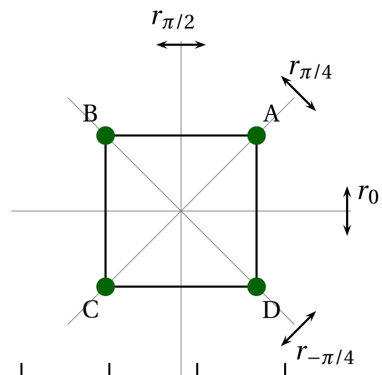
D_3	I	$R_{2\pi/3}$	$R_{-2\pi/3}$	r_0	$r_{\pi/3}$	$r_{-\pi/3}$
I	I	$R_{2\pi/3}$	$R_{-2\pi/3}$	r_0	$r_{\pi/3}$	$r_{-\pi/3}$
$R_{2\pi/3}$	$R_{2\pi/3}$	$R_{-2\pi/3}$	I	$r_{\pi/3}$	$r_{-\pi/3}$	r_0
$R_{-2\pi/3}$	$R_{-2\pi/3}$	I	$R_{2\pi/3}$	$r_{-\pi/3}$	r_0	$r_{\pi/3}$
r_0	r_0	$r_{-\pi/3}$	$r_{\pi/3}$	I	$R_{-2\pi/3}$	$R_{2\pi/3}$
$r_{\pi/3}$	$r_{\pi/3}$	r_0	$r_{-\pi/3}$	$R_{2\pi/3}$	I	$R_{-2\pi/3}$
$r_{-\pi/3}$	$r_{-\pi/3}$	$r_{\pi/3}$	r_0	$R_{-2\pi/3}$	$R_{2\pi/3}$	I

6. (There is more than one correct answer to this.) First use the multiplication table to write $r_{-\pi/3} = R_{2\pi/3} r_{\pi/3}$. And then write $R_{2\pi/3}$ using just the required two reflections: $R_{2\pi/3} = r_{\pi/3} r_0$. Thus, combining these we get

$$r_{-\pi/3} = R_{2\pi/3} r_{\pi/3} = (r_{\pi/3} r_0) r_{\pi/3} = r_{\pi/3} r_0 r_{\pi/3}.$$

(The other answer is $r_{-\pi/3} = R_{-2\pi/3} r_0 = (r_0 r_{\pi/3}) r_0 = r_0 r_{\pi/3} r_0$.)

Symmetry of a square



Recall that

$$\mathrm{D}_4 = \{I, R_{\pi/2}, R_{\pi}, R_{-\pi/2}, r_0, r_{\pi/4}, r_{\pi/2}, r_{-\pi/4}\}.$$

1. The multiplication table of D_4:

D_4	I	R	R^2	R^3	r_0	$r_{\pi/4}$	$r_{\pi/2}$	$r_{-\pi/4}$
I	I	R	R^2	R^3	r_0	$r_{\pi/4}$	$r_{\pi/2}$	$r_{-\pi/4}$
R	R	R^2	R^3	I	$r_{\pi/4}$	$r_{\pi/2}$	$r_{-\pi/4}$	r_0
R^2	R^2	R^3	I	R	$r_{\pi/2}$	$r_{-\pi/4}$	r_0	$r_{\pi/4}$
R^3	R^3	I	R	R^2	$r_{-\pi/4}$	r_0	$r_{\pi/4}$	$r_{\pi/2}$
r_0	r_0	$r_{-\pi/4}$	$r_{\pi/2}$	$r_{\pi/4}$	I	R^3	R^2	R
$r_{\pi/4}$	$r_{\pi/4}$	r_0	$r_{-\pi/4}$	$r_{\pi/2}$	R	I	R^3	R^2
$r_{\pi/2}$	$r_{\pi/2}$	$r_{\pi/4}$	r_0	$r_{-\pi/4}$	R^2	R	I	R^3
$r_{-\pi/4}$	$r_{-\pi/4}$	$r_{\pi/2}$	$r_{\pi/4}$	r_0	R^3	R^2	R	I

For brevity we have written $R = R_{\pi/2}$, and then $R^2 = R_\pi$ and $R^3 = R_{3\pi/2} = R_{-\pi/2}$.

NB: recall that products should be interpreted as acting from right to left: ab means apply b first and then a. This is because we're talking about composition of functions (that is, $ab = a \circ b$, or $ab(x) = a(b(x))$).

2. Compare, term by term, the elements in the r_0 row with the elements in the r_0 column. Whenever corresponding elements are equal, the 'multiplying element' commutes with r_0. For example, the third element of the column and row is $r_{\pi/2}$, and this implies $R^2 r_0 = r_0 R^2$. The answer is $C(r_0) = \{e, r_0, R_\pi, r_{\pi/2}\}$.

3. (a) The only elements that fix A are e and $r_{\pi/4}$, and these form a group (of course!) $G_A = \langle r_{\pi/4}\rangle$. Similarly $G_B = \langle r_{-\pi/4}\rangle$, and $G_C = G_A$, $G_D = G_B$.

4.

$$\rho(e) = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix} \qquad \rho(R_{\pi/2}) = \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$$

$$\rho(R_\pi) = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix} \qquad \rho(R_{\pi/2}) = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$$

$$\rho(r_0) = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} \qquad \rho(r_{\pi/4}) = \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}$$

$$\rho(r_{\pi/2}) = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix} \qquad \rho(r_{-\pi/4}) = \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$$

5. $G_{AC} = \{e, R_\pi, r_{\pi/4}, r_{-\pi/4}\}$. This forms a subgroup of order 4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (the group has order 4 and contains no element of order 4, so it can't be isomorphic to $\mathbb{Z}_4$). NB, this subgroup of D_4 is *not* conjugate to D_2.

6. For the suggested example (using the permutations given in question 3)

$$\begin{aligned}\rho(r_{\pi/4})\rho(r_0) &= \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix} \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} \\ &= \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}\end{aligned}$$

while $r_{\pi/4} r_0 = R_{\pi/2}$ (from multiplication table above) and

$$\rho(R_{\pi/2}) = \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$$

thus showing that $\rho(r_{\pi/4})\rho(r_0) = \rho(r_{\pi/4} r_0)$ as required.

7. Firstly, from the multiplication table one finds $r_{\pi/2} = r_{\pi/4} r_0 r_{\pi/4}$ (Indeed every element of D_4 can be written in terms of r_0 and $r_{\pi/4}$ — that is, they generate the group). On the other hand, the group generated by r_0 and $r_{\pi/2}$ is

$$H = \langle r_0, r_{\pi/2} \rangle = \{I, R_\pi, r_0, r_{\pi/2}\}.$$

(This group is called D_2, and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

8. Here we just go through the multiplication table: for each element different from e, R_π, we need only find one other element with which it does not commute. For example r_0 and $R_{\pi/2}$ do not commute, and nor do $r_{\pi/2}$ and $r_{\pi/4}$ etc.