Solutions for Chapter 6: Periodic Motion

Qu 6.1: Use the property of uniqueness of solutions of ODEs to show that if γ is a solution for which there is a $T > 0$ such that $\gamma(T) = \gamma(0)$ then $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$.

Solution Write the ODE as $\dot{\mathbf{x}} = f(\mathbf{x})$, and let $x_0 = \gamma(0)$. The solution after time t of the ODE with initial value x_0 is $\gamma(t)$. Now consider the function $\delta(t) = \gamma(t + T)$. First $\delta(0) = \gamma(T) = \gamma(0) = x_0$, and secondly

$$\delta'(t) = \gamma'(t + T) = f(\gamma(t + T)) = f(\delta(t)).$$

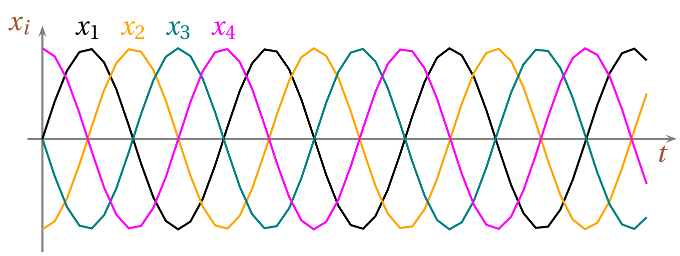
That is, $\delta(t)$ also satisfies the ODE, with the same initial condition as $\gamma(t)$ and therefore, by uniqueness of solutions, $\delta(t) = \gamma(t)$. That is, $\gamma(t + T) = \gamma(t)$, as required.

Qu 6.2: Consider a system of 4 identical coupled cells with symmetry D_4. Draw the cell diagram for such a system. Let $\gamma(t)$ be a periodic orbit with symmetry $\widetilde{\mathbb{Z}_4}$ generated by $((1\,2\,3\,4), \frac{1}{4}) \in S_4 \times S^1$ and period T. State the relation between the cells after a quarter of a period. If $x_1(t) = \sin(t)$ (with $T = 2\pi$), deduce the form of $x_j(t)$ for $j = 2, 3, 4$. Plot the graphs of the 4 functions $x_j(t)$, on the same diagram.

Solution Now the action of permutations imply $(1\,2\,3\,4) \cdot (x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3)$. Therefore, the spatio-temporal symmetry implies (recall $g \cdot \mathbf{x}(t) = \mathbf{x}(t + thetaT_0)$)

$$x_1(t) = x_2(t + \pi/2) \quad \text{etc.}$$

If $x_1(t) = \sin t$, then $x_2(t) = sin(t - \pi/2) = -\cos t$, $x_3(t) = sin(t - \pi) = -\sin t$ and $x_4(t) = \sin(t - 3\pi/2) = \cos t$.



Qu 6.3: Consider a system of 3 identical coupled cells with symmetry S_3. Draw the cell diagram for such a system. Let $\gamma(t)$ be a periodic orbit with symmetry $\widetilde{\mathbb{Z}_2}$ generated by $((1\ 2), \frac{1}{2}) \in S_3 \times S^1$ and period T. State the relation between the 3 cells after half a period, and deduce the period of cell 3.

Solution Write $\gamma(t) = (x_1(t), x_2(t), x_3(t))$. The fact that $((1\ 2), \frac{1}{2}) \in \Sigma_\gamma$, the symmetry group of γ, is equivalent to $\gamma(t + T/2) = (1\ 2)\gamma(t)$, which translates to

$$\begin{aligned}(x_1(t+T/2), x_2(t+T/2), x_3(t+T/2)) &= (1\ 2)(x_1(t), x_2(t), x_3(t)) \\ &= (x_2(t), x_1(t), x_3(t)).\end{aligned}$$

Since therefore $x_3(t + T/2) = x_3(t)$ it follows that x_3 has period $T/2$ (rather than T).

Qu 6.4: By considering the subgroups of $G = \mathsf{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and all possible homomorphisms from these to $S^1 = \mathbb{R}/\mathbb{Z}$, find all possible symmetry groups of periodic orbits in a system with D_2 symmetry. [Hint: there are 5 subgroups, and these have 1, 2, 2, 2 and 4 homomorphisms respectively giving 11 possible symmetry groups in all.]

Solution There are three copies of $\mathbb{Z}_2$ in D_2, one generated by r_0 another by $r_{\pi/2}$ and the third by R_π. Call these, $\mathsf{D}_1, \mathsf{D}_1'$ and C_2 respectively. Then the subgroups of D_2 are

$$\mathbb{1},\ \mathsf{D}_1,\ \mathsf{D}_1',\ \mathsf{C}_2,\ \mathsf{D}_2.$$

For each of these we need to consider all homomorphisms into S^1.

$\mathbb{1}$: Here there is only one homomorphism, namely $\phi(e) = 0$.

$\mathbb{Z}_2$: To take each of the order two subgroups together, let κ be their generator (so $\kappa = r_0$ in the first and R_π in the last). Let $\phi : \mathbb{Z}_2 \to S^1$ be a homomorphism. Since $\kappa^2 = e$ it follows that $2\phi(\kappa) = 0$:
[Recall that S^1 is written additively, so $\phi(\kappa^2) = 2\phi(\kappa)$).]
There are therefore two possible homomorphisms: $\phi(\kappa) = 0$ or $\phi(\kappa) = \frac{1}{2}$. The first leads to symmetry group $\mathbb{Z}_2$ and the second to $\widetilde{\mathbb{Z}_2}$. This should be applied to each of the subgroups of order 2.

D_2: Finally we take $H = \mathsf{D}_2 = \{I, r_0, r_{\pi/2}, R_\pi\}$, and let $\phi : \mathsf{D}_2 \to S^1$ be a homomorphism. Let κ be any of the order 2 elements. Then $\phi(\kappa) = 0$ or $\phi(\kappa) = \frac{1}{2}$ (like before). Note also that $\phi(\kappa_1\kappa_2) = \phi(\kappa_1)\phi(\kappa_2)$, and therefore if $\phi(\kappa_1) = \phi(\kappa_2) = \frac{1}{2}$ then $\phi(\kappa_1\kappa_2) = 0$. There are therefore 4 possibilities, listed below

	1	r_0	$r_{\pi/2}$	R_π
ϕ_0	0	0	0	0
ϕ_1	0	1/2	1/2	0
ϕ_2	0	1/2	0	1/2
ϕ_3	0	0	1/2	1/2

The first gives symmetry group D_2, the other three all versions of $\widetilde{\mathsf{D}_2}$, which we could call $\widetilde{\mathsf{D}_2}^1$, $\widetilde{\mathsf{D}_2}^2$ and $\widetilde{\mathsf{D}_2}^3$ respectively.

Conclusion: there are the following 11 possible symmetry groups:

$$\mathbb{1}, \mathsf{D}_1, \widetilde{\mathsf{D}_1}, \mathsf{D}_1', \widetilde{\mathsf{D}_1'}, \mathsf{C}_2, \widetilde{\mathsf{C}_2}, \mathsf{D}_2, \widetilde{\mathsf{D}_2}^1, \widetilde{\mathsf{D}_2}^2, \widetilde{\mathsf{D}_2}^3.$$

Qu 6.5: Repeat Example 6.10, but for the D_4 action on $\mathbb{R}^2$, showing the existence of periodic orbits with symmetries shown in Figure 6.1(b).

Solution Left to you ...

Qu 6.6: Find all 27 complex-axial symmetry groups of the action of $\mathbb{T}_d$ on $\mathbb{R}^3$ described in Section 4.5.

Solution The full solution is very lengthy: I will illustrate it with one case and some comments at the end. Begin by considering the subgroup $\mathbb{Z}_3$ generated by the matrix $R_d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (this rotates by $2\pi/3$ about the vertex $(1,1,1)$). This matrix has eigenvalues $1, e^{\pm i2\pi/3}$. The eigenvalue 1 has eigenvector $(1,1,1)$ so corresponds to the fixed point space in $\mathbb{C}^3$ where $x = y = z$, and these points are fixed by $\mathbb{Z}_3 < \mathbb{Z}_3 \times S^1$.
The eigenvalues $e^{\pm 2\pi i/3}$ are more interesting:

Qu 6.7: Suppose G acts on $\mathbb{R}^n$ and let X denote the space of all continuous maps $\gamma : S^1 \to \mathbb{R}^n$. There is an action of $G \times S^1$ on this space: if $\gamma \in X$ then define $(g,\theta)\cdot\gamma$ to be the map,

$$\big((g,\theta)\cdot\gamma\big)(t) = g\cdot\big(\gamma(t-\theta)\big).$$

Verify that this defines an action, and show that the stabilizer of an element γ is precisely its symmetry group Σ_γ.

Solution Let (g,θ) and $(h,\phi) \in G\times S^1$. We need to show the purported action satisfies

$$(g,\theta)\cdot\big((h,\phi)\cdot\gamma\big) = (gh,\theta+\phi)\cdot\gamma. \qquad (*)$$

To see this, first let $\delta = (h,\phi)\cdot\gamma$, which means $\delta(s) = h\cdot\gamma(s-\phi)$. Then the left-hand side of $(*)$ is

$$\begin{aligned}(g,\theta)\cdot\big((h,\phi)\cdot\gamma\big)(t) &= ((g,\theta)\cdot\delta)(t)\\ &= g\cdot\delta(t-\theta)\\ &= g\cdot(h\cdot\gamma(t-\theta-\phi)).\end{aligned}$$

On the other hand, the right-hand side of $(*)$ is

$$\begin{aligned}(gh,\theta+\phi)\cdot\gamma &= (gh)\cdot\gamma(t-(\theta+\phi)),\\ &= g\cdot(h\cdot\gamma(t-\theta-\phi))\end{aligned}$$

showing that indeed $(*)$ is correct. And since this holds for all (g,θ) and $(h,\phi)\in G\times S^1$, this is indeed an action.

For the second part, let $\gamma : S^1 \to \mathbb{R}^n$, and let $\Gamma = \big(G\times S^1\big)_\gamma$ be its stabilizer for this action. Let $(g,\theta)\in\Gamma$. Then, by the definition of a stabilizer,

$$\big((g,\theta)\cdot\gamma\big)(t) = \gamma(t).$$

Using the definition of the action, this becomes

$$g\cdot\gamma(t-\theta) = \gamma(t).$$

But this is precisely the definition of the elements of Σ_γ, so we are done.

