Solutions for Chapter 5: ODEs

Qu 5.1: Let $\mathbb{Z}_2 = \langle r \rangle$ act on $\mathbb{R}$ by $r \cdot x = -x$. Show that the differential equation $\dot{x} = \sin(2x)$ has symmetry group $\mathbb{Z}_2$. Let $x(t)$ be the solution with initial value $x(0) = 1$, and let $u(t)$ be the solution with initial value $u(0) = -1$. How are $x(t)$ and $u(t)$ related?

Solution The symmetry follows because sin is an odd function: $\sin(-2x) = -\sin(2x)$. Since $u(0) = -x(0)$ it follows that $u(t) = -x(t)$ (for all t in the domain of x).

Qu 5.2: Let D_3 be the usual dihedral subgroup of order 6 of $\mathsf{O}(2)$, generated by r_0 and $R_{2\pi/3}$. Consider the system of ODEs

$$\begin{cases} \dot{x} &= x + x^2 - y^2 \\ \dot{y} &= y - 2xy \end{cases}$$

(a) Show this system has D_3 symmetry.
(b) List all three axial subgroups of D_3. By choosing one of these, find all equilibria of this system with axial symmetry, and explain briefly why it is enough to consider only one of the axial subgroups.

Solution (a) Write this as $\dot{\mathbf{x}} = f(\mathbf{x})$ where $f(x,y) = \begin{pmatrix} x + x^2 - y^2 \\ y - 2xy \end{pmatrix}$. We need to show f is equivariant, for which it suffices to check $f(g \cdot \mathbf{x}) = g \cdot f(\mathbf{x})$ for each generator of the group. Now $\mathsf{D}_3 = \langle R_{2\pi/3}\, r_0 \rangle$, so we show this equivariant for $g = r_0$ and $g = R_{2\pi/3}$ in turn. There are two ways of doing this. The first is a lengthy calculations, and the second is a trick using complex numbers. If you like complex numbers, you can skip the first part.

Lengthy but straightforward calculation: First $g = r_0$. Now $r_0(x,y) = (x,-y)$, so

$$f(r_0(x,y)) = f(x,-y) = \begin{pmatrix} x + x^2 - y^2 \\ -y + 2xy \end{pmatrix}.$$

On the other hand

$$r_0 f(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x + x^2 - y^2 \\ y - 2xy \end{pmatrix} = \begin{pmatrix} x + x^2 - y^2 \\ -y + 2xy \end{pmatrix}.$$

Therefore $f(r_0 \cdot \mathbf{x}) = r_0 f(\mathbf{x})$.

Now consider $g = R_{2\pi/3} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (see Chapter 2). Then

$$\begin{aligned} f(R_{2\pi/3}\mathbf{x}) = f(-\tfrac{1}{2}x - \tfrac{\sqrt{3}}{2}y,\ \tfrac{\sqrt{3}}{2}x - \frac{1}{2}y) &= \begin{pmatrix} (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y) + (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)^2 - (\frac{\sqrt{3}}{2}x - \frac{1}{2}y)^2 \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y - 2(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)(\frac{\sqrt{3}}{2}x - \frac{1}{2}y) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}x - \frac{\sqrt{3}}{2}y - \frac{1}{2}x^2 + \sqrt{3}xy + \frac{1}{2}y^2 \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y + \frac{\sqrt{3}}{2}x^2 + xy - \frac{\sqrt{3}}{2}y^2 \end{pmatrix}, \end{aligned}$$

and we find $R_{2\pi/3} f(x,y)$ gives the same result. Thus f is indeed D_3 equivariant, and it follows that the system of ODEs has D_3 symmetry.

Calculation using complex numbers Treat $\mathbb{R}^2$ as $\mathbb{C}$ via $z = x + iy$, then f becomes $f(z) = z + \overline{z}^2$ (you can check), $r_0(z) = \overline{z}$ and $R_{2\pi/3}z = e^{2\pi i/3}z$. Equivariance then becomes checking that

$$f(\overline{z}) = \overline{f(z)}, \quad \text{and} \quad f(e^{2\pi i/3}z) = e^{2\pi i/3}f(z),$$

which is left to you.

(b) Recall that axial subgroups are those with 1-dimensional fixed point spaces. Since D_3 is acting on the plane, we are looking at lines of reflection. Since there are 3 lines of reflection, there are 3 axial subgroups:

$$H_1 = \langle r_0 \rangle, \quad H_2 = \langle r_{\pi/3} \rangle, \quad \text{and} \quad H_3 = \langle r_{-\pi/3} \rangle.$$

Choosing the axial subgroup H_1, we have $\mathrm{Fix}(H_1) = x$-axis. On the x-axis, $y = 0$ and the system becomes,

$$\dot{x} = x + x^2, \qquad \dot{y} = 0.$$

The equilibria occur when $x + x^2 = 0$, so when $x = 0$ and when $x = -1$. And since $y = 0$, these equilibria are at $(0,0)$ and $(-1,0)$.

The other equilibria with axial symmetry are obtained by rotation from the ones in $\mathrm{Fix}(H_1)$. There are therefore two other equilibria, at

$$\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \quad \text{and} \quad \left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right).$$

Rotation by $2\pi/3$ and $4\pi/3$ map one line of reflection to each of the two others, and the three equilibrium points outside the origin form a single orbit. This is closely related to the fact that the subgroups H_1, H_2 and H_3 are all conjugate (eg $H_2 = R_{2\pi/3}H_1R_{2\pi/3}^{-1}$, compare with Proposition 1.6).

Qu 5.3: Let L be an $n \times n$ matrix, and consider the first order ODE $\dot{\mathbf{x}} = L\mathbf{x}$ on $\mathbb{R}^n$. Show this is equivariant for a linear action of a group G if and only if L commutes with all the matrices in the representation of G.

Solution This is straightforward: we have $f(\mathbf{x}) = L\mathbf{x}$. Now, f is equivariant if and only if $\rho(g)f(\mathbf{x}) = f(\rho(g)\mathbf{x})$ (using ρ to make the representation explicit), for all $\mathbf{x}$ and all g. That is,

$$\rho(g)L\mathbf{x} = L\rho(g)\mathbf{x}, \quad (\forall \mathbf{x}, \forall g).$$

Since this equation holds for all $\mathbf{x}$ the two matrices $\rho(g)L$ and $L\rho(g)$ must be equal, which is to say that $\rho(g)$ and L commute (for all $g \in G$).

Qu 5.4: Consider the following family of system of ODEs in the plane

$$\begin{cases} \dot{x} &= ax + x^2 - y^2 \\ \dot{y} &= ay - 2xy \end{cases}$$

Here $a \in \mathbb{R}$ is a parameter. This is similar to a previous question, and the system has D_3 symmetry. Describe the bifurcations of equilibrium points that occur on the lines of symmetry as a is varied through $a = 0$.

Solution One of the lines of symmetry is the x-axis, or $y = 0$. On this line the differential equation becomes

$$\dot{x} = ax + x^2.$$

Equilibria occur when the right-hand side vanishes, so when $x(a + x) = 0$. For $a = 0$ there is only one equilibrium (at $x = 0$). For all $a \neq 0$ there are two equilibria: at $x = 0, -a$.

Qu 5.5: Consider the similar system with symmetry D_4:

$$\begin{cases} \dot{x} &=& x + x^3 - 3xy^2 \\ \dot{y} &=& y - 3x^2y + y^3. \end{cases}$$

(a) Show this system has D_4 symmetry.
(b) List all axial subgroups of D_4, and find all equilibria of this system with axial symmetry. (Explain briefly why in this case it is *not* enough to consider only one of the axial subgroups.)

Solution (a) We need to show that $f(x, y) = (x + x^3 - 3xy^2,\ y - 3x^2y + y^3)$ is equivariant. To this end, we show it satisfies $f(g(x, y)) = g(f(x, y))$ for g being each generator of D_4 (this is an easier calculation that for the D_3 equivariance of the previous question).
For $r_0(x, y) = (x, -y)$,

$$\begin{aligned} f(r_0(x, y)) = f(x, -y) &= (x + x^3 - 3x(-y)^2,\ (-y) - 3x^2(-y) + (-y)^3), \text{ while} \\ r_0(f(x, y)) &= (x + x^3 - 3xy^2,\ -y + 3x^2y - y^3). \end{aligned}$$

and these are equal.
For $g = R_{\pi/2}$, recall that $R_{\pi/2}(x, y) = (-y, x)$, and so

$$\begin{aligned} f(R_{\pi/2}(x, y)) = f(-y, x) &= ((-y) + (-y)^3 - 3(-y)x^2,\ x - 3(-y)^2x + x^3) \\ R_{\pi/2}(f(x, y)) &= (-(y - 3x^2y + y^3),\ x + x^3 - 3xy^2) \end{aligned}$$

and again, these are equal. It now follows that the system of ODEs has D_4 symmetry.
There is also a proof using complex numbers similar to the argument of Problem 5.2. In this case $f(z) = z + \overline{z}^3$.
(b) Recall that axial subgroups are those with 1-dimensional fixed point spaces. Since D_4 is acting on the plane, we are looking at lines of reflection. There are 4 lines of reflection, and hence there are 4 axial subgroups:

$$H_1 = \langle r_0 \rangle, \quad H_2 = \langle r_{\pi/2} \rangle, \quad H_3 = \langle r_{\pi/4} \rangle, \quad \text{and} \quad H_4 = \langle r_{-\pi/4} \rangle.$$

Note that H_1 and H_2 are conjugate, and H_3 and H_4 are conjugate. Choosing the axial subgroup H_1, we have $\mathrm{Fix}(H_1) = x$-axis. On the x-axis, $y = 0$ and the system becomes,

$$\dot{x} = x + x^3, \qquad \dot{y} = 0.$$

The only equilibrium on the x-axis is therefore the origin.
Now consider H_3. The fixed point space is the line $y = x$. Substituting this into the ODE gives

$$\dot{x} = x - 2x^3.$$

The equilibria therefore occur at $x = y = \pm 1/\sqrt{2}$ (and the origin).
Conclusion: the axial equilibria are the origin and the 4 vertices of a square:

$$(0,0), \quad \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), \quad \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right).$$

Qu 5.6: Check that the two systems in Examples 5.4 have symmetry S_3 and $\mathbb{Z}_4$ respectively.

Solution Left to you …

Qu 5.7: Consider the following system of ordinary differential equations,

$$\begin{cases} \dot{x} &= -x + yz^2 \\ \dot{y} &= -y + xz^2 \\ \dot{z} &= z(1 + xy - z^2). \end{cases} \qquad (*)$$

Consider the action of the group $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i). Show that the matrices A, B, C do indeed generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (you need to show that $A^2 = I$ etc, and A, B, C all commute).

(ii). Show that the system $(*)$ has symmetry G.

(iii). Deduce that the x-y plane and the z-axis are each invariant under the evolution of the system, stating carefully any results used.

(iv). Can you find other invariant subspaces?

(v). Find all the equilibrium points that lie on these subspaces.

(vi). Find the unique solution to this system with initial value $(x, y, z) = (1, 1, 0)$. What is the limit as $t \to \infty$ of this solution?

Solution

(i). This is just a calculation; show $A^2 = I$, $AB = BA$ etc.

(ii). It is enough to check equivariance for the three generators A, B and C. Let

$$f(x, y, z) = \left(-x + yz^2,\ -y + xz^2,\ z(1 + xy - z^2)\right).$$

We want to check f is equivariant. First with A:

$$\begin{aligned} f(A(x, y, z)) &= f(-x, -y, z) = (x - yz^2,\ y - xz^2,\ z(1 + xy - z^2))^T, \\ Af(x, y, z) &= (-(-x + yz^2),\ -(-y + xz^2),\ z(1 + xy - z^2))^T, \end{aligned}$$

and these are equal. A similar calculation can be done for B and C.

(iii). The x-y plane is fixed by $\mathbb{Z}_2^B := \langle B\rangle$, while the z-axis is fixed by the subgroup $\langle A, C\rangle$ of order 4. By the conservation of symmetry theorem [fill in the statement...], both are invariant under the evolution of the system.

(iv). There are several other fixed point spaces:

subgroup	fixed point space
G	$\{0\}$
$\mathbb{Z}_2^C := \langle C\rangle$	the plane $x = y$
$\langle B, C\rangle$	the line $y = x$, $z = 0$
$\langle AC\rangle$	the plane $y = -x$
$\langle AC, B\rangle$	the line $y = -x$, $z = 0$

(v). For example, for $\mathrm{Fix}(\mathbb{Z}_2^B)$ (defined above), we can put $z = 0$ into the equation to see that the equilibria occur at $-x = 0, -y = 0$, so at the origin only. On $\mathrm{Fix}(\langle A, C\rangle)$ (the z-axis), we put $x = y = 0$ and have $z(1 - z^2) = 0$, so giving three equilibria: $(0, 0, 0)$ (which we already know), $(0, 0, 1)$ and $(0, 0, -1)$. The others are as follows:

subgroup	equilibria
G	$(0, 0, 0)$
$\mathbb{Z}_2^C := \langle C\rangle$	(0,0,0), $(0, 0, 1)$ and $(0, 0, -1)$
$\langle B, C\rangle$	$(0, 0, 0)$
$\langle AC\rangle$	(0,0,0), $(0, 0, 1)$ and $(0, 0, -1)$
$\langle AC, B\rangle$	(0,0,0).

(vi). This initial point $\mathbf{x}_0 = (1, 1, 0)$ lies in $\mathrm{Fix}(H)$ where $H = \langle B, C\rangle$. The solution will therefore (by the conservation of symmetry theorem) lie entirely in this subspace, so will satisfy $x = y$ and $z = 0$ throughout. We therefore only need solve the single equation

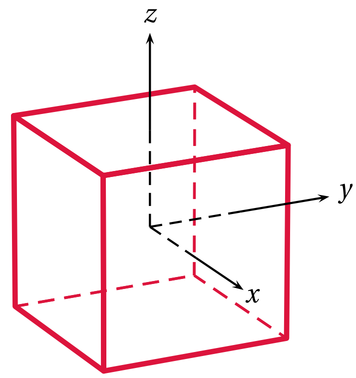
$$\dot{x} = -x$$

(putting $z = 0$ in the first equation), and with initial value $x_0 = 1$. This has solution $x(t) = e^{-t}$. The solution is therefore

$$\gamma(t) = (x(t), y(t), z(t)) = (e^{-t}, e^{-t}, 0).$$

As $t \to \infty$ so $\gamma(t) \to (0, 0, 0)$.

Qu 5.8: The octahedral group $\mathbb{O}_h$ is the group of all symmetries of the cube (including reflections). With vertices at the 8 points $(\pm 1, \pm 1, \pm 1)$, it is generated as follows.



Generators:

$$R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_d = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$r_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Here R_z is a rotation about the z-axis by $\pi/2$, R_d is a rotation by $2\pi/3$ about the diagonal $x = y = z$, and r_z is the reflection in the x-y plane.
Consider the following family of potential functions in 3-D:

$$V = \lambda\left(x^2 + y^2 + z^2\right) - 2\left(x^4 + y^4 + z^4\right) + 3\left(x^2y^2 + z^2x^2 + z^2y^2\right),$$

(this is an approximation to the system of 8 identical springs each attached to the vertex of a cube, and all attached to a common particle).
(i) Show V has symmetry $\mathbb{O}_h$ (it is enough to show it is invariant under the 3 given generators).
(ii) Show that the lines $L_1 = \{(0,0,z) \mid z \in \mathbb{R}\}$ and $L_2 = \{(x,x,0) \mid x \in \mathbb{R}\}$, and $L_3 = \{(x,x,x) \mid x \in \mathbb{R}\}$, are all 1-dimensional fixed point spaces, and find the corresponding axial subgroups. [Hint: sketch each of these lines on the figure with the cube.]
(iii) Find critical points (equilibria) occurring in these 1-dimensional fixed-point subspaces, and describe how these appear/disappear as λ varies (i.e., the bifurcations involved).
(iv) Find the other 1-dimensional fixed point spaces (all others are equivalent under the summery group $\mathbb{O}_h$ to L_1, L_2 or L_3), and list the corresponding equilibrium points.

Solution (i) Just a check!
(iii) Each point of the z-axis L_1 is fixed by rotations about the z-axis, so by R_z, also by those reflections preserving the z-axis, such as $(x, y, z) \mapsto (-x, y, z)$. Looking down the z-axis at the cube, the z-axis will map to the origin and the symmetry group will be D_4. The matrices are the 3×3 matrices of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$$

with $A \in \mathsf{D}_4$. It's important to note not just that the points o f the z-axis are fixed by this subgroup, but that they are the only points fixed by this subgroup.
Each point of L_2 is fixed by the rotation by π about the line $x = y$, $z = 0$ as well as the reflection r_z in the xy-plane, giving a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
For L_3: the points on the diagonal are fixed by the rotation R_d, which generates a subgroup of order 3 of $\mathbb{O}_h$. There are also some reflections exchanging vertices adjacent to $(1, 1, 1)$. Indeed, looking down the diagonal of the cube, we see a 3-fold (i.e., triangular) symmetry, and the symmetry group is a subgroup isomorphic to D_3, generated by the rotation R_d and a reflection such as $(x, y, z) \mapsto (y, x, z)$.
(iii) V always has a critical point at the origin (either by direct calculation, or because $\mathrm{Fix}(\mathbb{O}_h, \mathbb{R}^3) = \{0\}$). The Hessian of V at the origin is $2\lambda I_3$. Thus, V has a local minimum there when $\lambda > 0$ and a local maximum when $\lambda < 0$. When $\lambda = 0$ the critical point is degenerate, and bifurcations may occur.
On the z-axis (or similarly on the other coordinate axes) we have

$$V(0, 0, z) = \lambda z^2 - 2z^4$$

and the critical points occur at $z = 0$ (the origin) and $z^2 = \lambda/4$, so at

$$z = \pm\tfrac{1}{2}\sqrt{\lambda}.$$

These occur for $\lambda > 0$, and form a pitchfork bifurcation. Identical bifurcations occur along the other axes (illustrating the importance of conjugate subgroups).
On the axes through the vertices, we have, for example, $x = y = z$. Then

$$V(x, x, x) = 3\lambda x^2 + 3x^4.$$

Apart from the origin, the critical points occur when $\lambda - 2x^2 = 0$, and so at

$$x = \pm\sqrt{-\lambda/2}.$$

This (in contrast to the previous case) is real when $\lambda < 0$.
Finally, for the axes through the midpoints of opposite edges: one such axis is in the x-y plane, with $y = x$:

$$V(x, x, 0) = 2\lambda x^2 - x^4.$$

This time, apart from the origin, the critical points occur when $\lambda - x^2 = 0$, so here

$$x = \pm\sqrt{\lambda}$$

Similarly to the first case, these are real when $\lambda > 0$.
All three are pitchfork bifurcations.

The full bifurcation diagram is: