Solutions for Chapter 4: Symmetry princple

Qu 4.1: Let $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition modulo 3, and let $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ (note that $\omega^3 = 1$). Consider the action of $\mathbb{Z}_3$ on the complex plane $\mathbb{C}$ defined by

$$n \cdot z = \omega^n z$$

(i) Show first this is indeed an action. (ii) Show that the equation $z^3 = 8$ has symmetry $\mathbb{Z}_3$ and that the set of solutions also has this symmetry.

Solution (i) Just need to check that the action satisfies the homomorphism property. Let $m, n \in \mathbb{Z}_3$, then

$$m \cdot (n \cdot z) = \omega^m(\omega^n z) = \omega^{n+m} z = (m+n) \cdot z$$

as required.
(ii) If we transform z by $n \in \mathbb{Z}_3$ the equation becomes $(n \cdot z)^3 = 1$. But it's easy to see that $(n \cdot z)^3 = z^3$, so that works fine. The set of solutions of the equation is $\{2, 2\mathrm{e}^{2\pi\mathrm{i}/3}, 2\mathrm{e}^{-2\pi\mathrm{i}m/3}\}$, and multiplying any of these elements by ω^m gives another element in the set, showing that this set indeed has $\mathbb{Z}_3$ symmetry.

Qu 4.2: Let the group G act on two sets X and Y, and suppose that $\phi : X \to Y$ is equivariant. If in addition we suppose ϕ is a bijection, show that $\phi^{-1} : Y \to X$ is also equivariant.

Solution ϕ^{-1} can be defiined uniquely by the condition that $\phi^{-1}(\phi(x)) = x$ for all $x \in X$. Then, for each $x \in X$,

$$\phi^{-1}(g \cdot \phi(x)) = \phi^{-1}(\phi(g \cdot x)) = g \cdot x = g \cdot \phi^{-1}(\phi(x))$$

[Equivalently, we can write $y = \phi(x)$, then we have shown that $\phi(g \cdot y) = g \cdot \phi^{-1}(y)$, for all $y \in Y$.]

Qu 4.3: Let V be a representation of G, and let $A : V \to V$ be a linear map (a matrix), which is equivariant. Recall that if $\mathbf{v} \neq \mathbf{0}$ satisfies $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$ one says $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.

(i). Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then so is $g \cdot \mathbf{v}$ for each $g \in G$.

(ii). Let E_λ be the λ-eigenspace of A,

$$E_\lambda = \{\mathbf{v} \in V \mid A\mathbf{u} = \lambda\mathbf{u}\}.$$

Show that E_λ is G-invariant.

(iii). Let G_λ be the *generalized eigenspace* of A:

$$G_\lambda = \{\mathbf{v} \in V \mid (A - \lambda I)^n \mathbf{v} = \mathbf{0}\},$$

where $n = \dim V$. Show that G_λ is also G-invariant.

Solution (i) Suppose $A\mathbf{v} = \lambda\mathbf{v}$. Then $A(g\cdot\mathbf{v}) = (A\rho(g))\mathbf{v} = \rho(g)A\mathbf{v} = \rho(g)\lambda\mathbf{v} = \lambda g\cdot\mathbf{v}$. Here $\rho(g)$ is the matrix representing the action of g (you can also write this with g in place of $\rho(g)$, assuming g is itself a matrix.)
(ii) This is the same as (1), written differently.
(iii) Let $\mathbf{v}\in G_\lambda$; that is, $(A-\lambda I)^n\mathbf{v} = \mathbf{0}$. If we show that $g(A-\lambda I) = (A-\lambda I)g$ then we are done, for (by induction) $g(A-\lambda I)^n = (A-\lambda I)^n g$, and

$$(A-\lambda I)^n g\mathbf{v} = g(A-\lambda I)^n\mathbf{v} = g\mathbf{0} = \mathbf{0}.$$

So, why is $g(A-\lambda I) = (A-\lambda I)g$? This is easy: we're given that A commutes with g, and I commutes with everything, hence

$$g(A-\lambda I) = (gA - g\lambda I) = (Ag - \lambda I g) = (A-\lambda I)g.$$

Qu 4.4: Consider the function $f(x,y) = x^2+y^2-x^4-y^4$. Show this is invariant under the group D_4 and find its set $C(f)$ of critical points. Describe how the group acts on this set (i.e., determine the orbits and the orbit type for each orbit), and hence state the Burnside type of the action on $C(f)$.

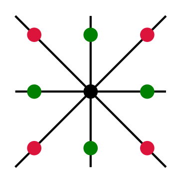
Solution Now D_4 has two generators, and we can take either $\mathsf{D}_4 = \langle r_0,\, r_{\pi/4}\rangle$ or $\mathsf{D}_4 = \langle r_0,\, R_{\pi/2}\rangle$. Let us use the former to check invariance (recall: it suffices to check for invariance under a set of generators of the group). Now $f(r_0(x,y)) = f(x,-y) = f(x,y)$ (as f is even in y). Similarly, $f(r_{\pi/4}(x,y)) = f(y,x) = f(x,y)$ (since f is symmetric in x and y).
Now to find its critical points:

$$f_x = 2x(1-2x^2), \quad f_y = 2y(1-2y^2).$$

The set of solutions $C(f)$ of these equations is (written as a union of orbits)

$$C(f) \;=\; \big\{(0,0)\big\}\cup\big\{(\tau,0),\,(-\tau,0),\,(0,\tau),\,(0,-\tau)\big\}\cup\big\{(\tau,\tau),\,(-\tau,\tau),\,(\tau,-\tau),\,(-\tau,-\tau)\big\}$$

where $\tau = 1/\sqrt{2}$. There are therefore three orbits. The first has stabilizer D_4. The second includes the point $(\tau,0)$ which has stabilizer D_1, while the third has stabilizer a conjugate copy of D_1 (conjugate in $\mathsf{O}(2)$ but not in D_4), call this D_1'. Then the orbit types are (D_4), (D_1) and (D_1'). There is one of each type, so the Burnside type is therefore



$$1(\mathsf{D}_4) \;+\; 1(\mathsf{D}_1) \;+\; 1(\mathsf{D}_1').$$

Qu 4.5: Let V be a representation of G with $V^G = \{0\}$. Prove directly that if $f : V \to \mathbb{R}$ is an invariant function then it has a critical point at 0. [By directly, I mean do not use the Principle of Symmetric Criticality, but you may use its proof to inspire you.]

Solution We have proved that since f is invariant, grad $f : V \to V$ is equivariant. We have also proved that if $\phi : V \to V$ is equivariant, and $y = \phi(x)$ then $G_x \leq G_y$.
Putting these together, shows that grad $f(0) \in V^G$ (because if $y = \text{grad} f(0)$ then $G_y \geq G_0 = G$).
But $V^G = \{0\}$ and hence grad $f(0) = 0$ as required.

Qu 4.6: Find all the critical points of the D_3-invariant function $f(x,y) = \lambda(x^2+y^2) + \frac{1}{3}x^3 - xy^2$. Relate these to the fixed point subspaces for different subgroups of D_3 (refer to Fig. 4.1).

Solution The critical points are given by $f_x = f_y = 0$ (partial derivatives). Here

$$f_x = 2\lambda x + x^2 - y^2, \quad f_y = 2y(\lambda - x).$$

To solve $f_x = f_y = 0$, first consider $f_y = 0$. This implies $y = 0$ or $x = \lambda$. If $y = 0$ then $f_x = 0$ shows $x = 0$ or $x = -2\lambda$. This gives two points with $y = 0$, namely $(0,0)$ and $(-2\lambda, 0)$. If on the other hand, $y \neq 0$ then $x = \lambda$ and $f_x = 0$ implies $y = \pm\sqrt{3}\,\lambda$. These are two other points: $(\lambda, \pm\sqrt{3}\,\lambda)$. There are thus 4 critical points (although if $\lambda = 0$ these four points are all the same!).
The relation to symmetry is: $(0,0)$ is fixed by the whole group D_3, next $(-2\lambda, 0)$ is in the line of reflection of r_0. The other two points are in the lines of reflection of $r_{\pm\pi/3}$. These each have symmetry $\mathbb{Z}_2$ for different copies of $\mathbb{Z}_2$ in D_3.
A little more thought: the origin is one orbit of critical points, and the other three points form another orbit. Since they are in the same orbit, their stabilizers are conjugate (Proposition 1.5). In Fig. 4.1, which is for a very similar function, there are also 4 critical points: the origin is one (it's a local minimum) and the other 3 are saddle points at the crossings of the light blue 'curve'.

Qu 4.7: For the system of 4 springs discussed in lectures (Example 4.12), study the critical points in the subspace $\mathrm{Fix}(K,\mathbb{R}^2)$, where $K = \langle r_{\pi/4}\rangle$.

Solution The points (x,y) fixed by $r_{\pi/4}$ are the points on the diagonal $y = x$. The potential energy V on this line is given by substituting $y = x$ in the expression on p.4.6:

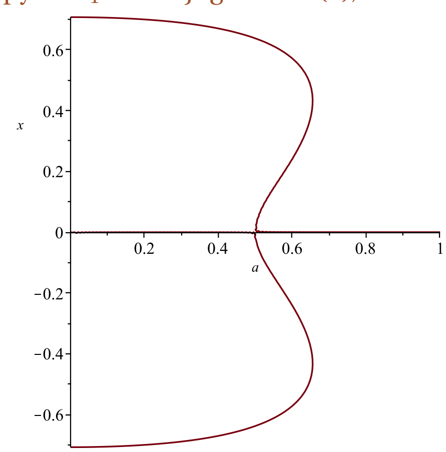
$$V(x,x) = \left(\sqrt{(x-a)^2+x^2}-1\right)^2 + \left(\sqrt{(x+a)^2+x^2}-1\right)^2.$$

Rather than analyze this (which is possible—just complicated), we follow Example 4.12 and consider the Taylor series at 0:

$$V(x,x) = 2(a-1)^2 + 2(2-a^{-1})x^2 - \frac{3}{2a^3}x^4 + O(5)$$

Like in the example, this has a local minimum at 0 when $a > 1/2$ and a local max when $a < 1/2$. The other critical points are at $x = \pm\frac{\sqrt{6}}{3}\sqrt{2a-1}$. If $a < 1/2$ the origin is the only critical point, but as a then increases through $a = 1/2$ two new critical points appear in a pitchfork bifurcation. These new critical points have less symmetry than the origin (or than the problem as a whole), namely $\mathsf{D}_1' = \langle r_{\pi/4}\rangle$ (a conjugate copy of D_1 — conjugate in $\mathsf{O}(2)$, not in D_4).

On the right is the diagram of all critical points on the diagonal $y = x$ for a in the range $[0,1]$ (produced numerically by computer). If you focus on a neighbourhood of $(a,x) = (1/2,0)$ you see a pitchfork bifurcation.



Qu 4.8: Let G act on a set X, and let Ω be the set of all functions $f : X \to \mathbb{R}$. Show that the following formula defines an action of G on Ω:

$$(g \cdot f)(x) = f(g^{-1}x), \quad \text{for } f \in \Omega,\ g \in G,\ x \in X.$$

In other words, $g \cdot f = f \circ g^{-1}$.
[Note: The inverse here should be reminiscent of the action by right multiplication of a group on itself, from Chapter 1 (§1.3) which also involves an inverse.]

Solution We need to show $g \cdot (h \cdot f) = gh \cdot f$. Now,

$$gh \cdot f = f \circ (gh)^{-1} = f \circ (h^{-1} \circ g^{-1}) = (f \circ h^{-1}) \circ g^{-1} = (h \cdot f) \circ g^{-1} = g \cdot (h \cdot f)$$

as required. Note we are using the fact that composition of functions is associative.

Qu 4.9: Suppose V, W are representations of a group G. Let $\phi_j : V \to W$ be two equivariant maps, and let $f_j : V \to \mathbb{R}$ be two invariant functions ($j = 1,2$). Show that the map $\psi : V \to W$ given by

$$\psi(\mathbf{v}) = f_1(\mathbf{v})\,\phi_1(\mathbf{v}) + f_2(\mathbf{v})\,\phi_2(\mathbf{v})$$

is equivariant.

Solution This is a straightforward calculation: let $g \in G$ and $\mathbf{v} \in V$. Then

$$\begin{aligned}
\psi(g\cdot\mathbf{v}) &= f_1(g\cdot\mathbf{v})\,\phi_1(g\cdot\mathbf{v}) + f_2(g\cdot\mathbf{v})\,\phi_2(g\cdot\mathbf{v}) \quad \text{(definition)} \\
&= f_1(\mathbf{v})(g\cdot\phi_1(\mathbf{v})) + f_2(\mathbf{v})\,(g\cdot\phi_2(\mathbf{v})) \quad \text{(invariance/equivariance)} \\
&= g\cdot\big(f_1(\mathbf{v})\,\phi_1(\mathbf{v}) + f_2(\mathbf{v})\,\phi_2(\mathbf{v})\big) \quad \text{(linearity)} \\
&= g\cdot\psi(\mathbf{v}). \quad \text{(definition)}
\end{aligned}$$

The penultimate equality (marked linearity) holds because the action of g is linear, so

$$g\cdot(a\mathbf{u} + b\mathbf{v}) = a(g\cdot\mathbf{u}) + b(g\cdot\mathbf{v})$$

for any scalars a, b, and here $f_1(\mathbf{v})$ and $f_2(\mathbf{v})$ are both scalars (real numbers).
[*Comment for more advanced algebraists: this shows that the set of equivariant maps is a module over the ring of invariant functions.*]

Qu 4.10: Find all homomorphisms of the cyclic group $\mathbb{Z}_4$ to the cyclic group $\mathbb{Z}_6$. [Hint: If H is a cyclic group generated by a, and $\phi : H \to G$ a homomorphism, then ϕ is entirely determined by knowing $\phi(a)$.]

Solution Write the groups additively (addition modulo 4 and 6 respectively). The first group $\mathbb{Z}_4$ is generated by 1, so we need only state the value of $\phi(1)$. It is of course an element of $\mathbb{Z}_6$, and we consider case by case (recall that if $\phi : H \to G$ is a HM then $\phi(e_H) = e_G$, or in this case $\phi(0) = 0$). The only problem arises because we need $\phi(0) = \phi(4)$ (since $0 = 4$ in $\mathbb{Z}_4$) and $\phi(4) = 4\phi(1)$).

- $\phi(1) = 0$: This is the trivial HM $\phi(a) = 0$ for all $a \in \mathbb{Z}_4$.
- $\phi(1) = 1$: This does not define a HM, since $\phi(4) = 4\phi(1) = 4 \neq 0$ in $\mathbb{Z}_6$.
- $\phi(1) = 2$: $\phi(0) = \phi(4) = 4\phi(1) = 8 \neq 0$, so not a HM.
- $\phi(1) = 3$: $\phi(0) = \phi(4) = 4\phi(1) = 12 = 0$ (in $\mathbb{Z}_6$), so this is a HM.
- $\phi(1) = 4$ and 5: these are also not homomorphism for similar reasons.

There are therefore only two homomorphisms $\mathbb{Z}_4$ to $\mathbb{Z}_6$, one of which is trivial and the other has image $\{0, 3\} \subset \mathbb{Z}_6$.

