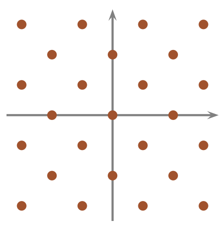
Solutions for Chapter 3: Lattices & Wallpaper Groups

Qu 3.1: Sketch the points (x, y) of the lattice

$$L = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in \mathbb{Z},\ x + y \in 2\mathbb{Z}\},$$

in the range $-3 \le x \le 3$ and $-3 \le y \le 3$. Show that L is generated by $(1,1)$ and $(2,0)$.

Solution
First the diagram:



The lattice L

For the second part, write $\mathbf{u} = (1,1)$ and $\mathbf{v} = (2,0)$ (these should be column vectors, but I'll use row vectors here to save space), and write $L' = \mathbb{Z}\{\mathbf{u},\mathbf{v}\}$. We want to show $L' = L$.
Now any element of L' can be written $\mathbf{w} = m\mathbf{u} + n\mathbf{v}$ with $m, n \in \mathbb{Z}$. Then $\mathbf{w} = (m + 2n, m)$ which clearly satisfies the conditions for being an element of L. Thus, $L' \subset L$.
For the converse, let $(x, y) \in L$. Then there are $a, b \in \mathbb{R}$ such that $(x, y) = a\mathbf{u} + b\mathbf{v} = (a+2b, a)$ (since $\{\mathbf{u},\mathbf{v}\}$ is a basis for $\mathbb{R}^2$). There remains to show $a, b \in \mathbb{Z}$, for then $L \subset L'$. Now, since $(x, y) \in L$, it follows that $y = a \in \mathbb{Z}$. Then $x + y = 2(a + b) \in 2\mathbb{Z}$ implies $b \in \mathbb{Z}$ (why?), so we are done.

Qu 3.2: Consider the lattice $L = \mathbb{Z}^2$. Show that L can be generated by the vectors $\mathbf{a} = (7,3)$ and $\mathbf{b} = (9,4)$.

Solution Hints: First write $L' = \mathbb{Z}\{\mathbf{a},\mathbf{b}\}$. We want to show $L' = L$. Easy to check $L' \subset L$ (every element of L' has integer coordinates), and it remains to show the converse.
Do this by showing that $(1,0) \in L'$ and $(0,1) \in L'$. That is, there are integers $m, n \in \mathbb{Z}$ such that $(1,0) = m\mathbf{a} + n\mathbf{b}$, and similarly for $(0,1)$. Since L is generated by $(1,0)$ and $(0,1)$ it follows that $L \subset L'$.

Qu 3.3: Extending the previous problem, show that $L = \mathbb{Z}^2$ is generated by integer vectors (a, b) and (c, d) whenever $ad - bc = \pm 1$. [Hint: Consider the matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and show the inverse matrix has integer entries iff $\det A = \pm 1$.]

Solution See coursework.

Qu 3.4: Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Show that they are orthogonal if and only if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

Solution We show the condition is equivalent to $\mathbf{a}\cdot\mathbf{b} = 0$, which, for non-zero vectors, means they are orthogonal.

$$\begin{aligned} |\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \quad &\iff \quad |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2 \\ &\iff \quad |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \\ &\iff \quad \mathbf{a}\cdot\mathbf{b} = 0, \end{aligned}$$

[NB: to expand the first line, use $|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}$.]

Qu 3.5: Let $L = \left\{(x,y)\in\mathbb{R}^2 \mid y\in\mathbb{Z}, \sqrt{2}(x-y)\in\mathbb{Z}\right\}$. First show L is a subgroup of $\mathbb{R}^2$ (under vector addition). Second show that

$$L = \left\{ \begin{pmatrix} a+\frac{1}{\sqrt{2}}b \\ a \end{pmatrix} \in \mathbb{R}^2 \mid a,b\in\mathbb{Z} \right\},$$

and hence find two generators of L and deduce that it is a lattice.

Solution Let $(x_1,y_1)\in L$ and $(x_2,y_2)\in L$. We want to show $(x_1,y_1)-(x_2,y_2)\in L$ (subgroup criterion). Now $y_1,y_2\in\mathbb{Z}$ implies $y_1-y_2\in\mathbb{Z}$. ALso

$$\sqrt{2}\big((x_1-x_2)-(y_1-y_2)\big) = \sqrt{2}(x_1-y_1)-\sqrt{2}(x_2-y_2)$$

and this is the difference between two integers (since we are assuming $\sqrt{2}(x_1-y_1)\in\mathbb{Z}$ and $\sqrt{2}(x_2-y_2)\in\mathbb{Z}$) and so it too is an integer. It follows that indeed $(x_1,y_1)-(x_2,y_2)\in L$.
Now let

$$L' = \left\{ \begin{pmatrix} a+\frac{1}{\sqrt{2}}b \\ a \end{pmatrix} \in \mathbb{R}^2 \mid a,b\in\mathbb{Z} \right\},$$

We want to show $L = L'$. Let $(x,y)\in L'$. Then $x = a+b\sqrt{2}$ and $y = a$ (with $a,b\in\mathbb{Z}$). Therefore $y = a\in\mathbb{Z}$ and $\sqrt{2}(x-y) = \sqrt{2}(a+\frac{1}{\sqrt{2}}b-a) = b\in\mathbb{Z}$. Thus, $(x,y)\in L$ and so $L'\subset L$. For the converse inclusion, let $(x,y)\in L$. We can find unique $a,b\in\mathbb{R}$ such that $x = a+\frac{1}{\sqrt{2}}b$ and $y = a$ (simultaneous equations). We get $a = y$ and $b = \sqrt{2}(x-y)$. Since $(x,y)\in L$ it follows that $a,b\in\mathbb{Z}$ and hence $(x,y)\in L'$. Thus $L\subset L'$ and therefore $L = L'$.
Generators for L' (and hence for L) are easy to find: for one put $a = 1, b = 0$ and for the other $b = 1, a = 0$. Two generators are therefore,

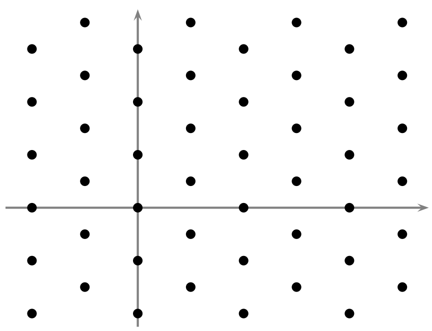
$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Many others are also possible, for example $\{-\mathbf{u},\mathbf{v}\}$ or $\{\mathbf{u}+\mathbf{v},-\mathbf{v}\}$, or ...

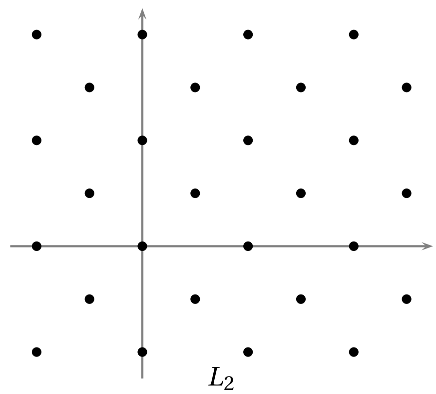
Qu 3.6: Consider the two lattices in $\mathbb{R}^2$ defined by,

$$L_1 = \{(2m+n, \tfrac{1}{2}n) \mid m,n \in \mathbb{Z}\} \quad \text{and} \quad L_2 = \{(2m+n, n) \mid m,n \in \mathbb{Z}\}$$

shown in the figures below. In each case, determine vectors **a**, **b** according to the conventions, and find the point group. Describe how the point group acts on the lattice. Which of the 5 types of lattice is each of these?



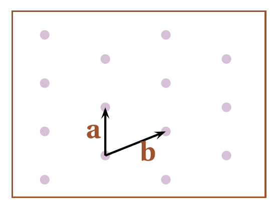
L_1



L_2

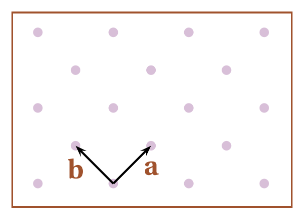
Solution

In the diagrams on the right, **a** and **b** generate the lattice of translations. To see this, note that the following vectors are in the respective lattice:



$$\text{in } L_1: \begin{pmatrix}2\\0\end{pmatrix},\ \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix}, \quad \text{and in } L_2: \begin{pmatrix}2\\0\end{pmatrix},\ \begin{pmatrix}1\\1\end{pmatrix},\ \begin{pmatrix}0\\2\end{pmatrix}.$$

This helps find the shortest vector **a**. For L_1 it's $(0,1)$, and then $\mathbf{b} = (1, \frac{1}{2})$ (as column vectors of course!) For L_2 the shortest is (eg) $\mathbf{a} = (1,1)$ [or we could take $\mathbf{a} = (1,-1)$]. Now the shortest vector linearly independent of **a** is $\mathbf{b} = (-1,1)$ (or $\mathbf{b} = (1,-1)$ — usually we pick the one for which $|\mathbf{a}-\mathbf{b}| < |\mathbf{a}+\mathbf{b}|$, but here these are equal). Thus we have,



for L_1: $|\mathbf{a}| < |\mathbf{b}| = |\mathbf{a}-\mathbf{b}| < |\mathbf{a}+\mathbf{b}|$ — a centred rectangular lattice, while
for L_2: $|\mathbf{a}| = |\mathbf{b}| < |\mathbf{a}-\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$ — a square lattice.

For the point group J:

- Each point of the lattice is a centre of rotation by π, and hence $R_\pi \in J$;
- there is a reflection in the horizontal axis, and hence $r_0 \in J$;
- there is also a reflection in the y-axis, and hence $r_{\pi/2} \in J$
- for the first lattice there is no centre of rotation with an angle less than π, so that is all the point group. For the second, $R_{\pi/4} \in J$.

Hence, for the first, $J = \mathsf{D}_2 = \{I, R_\pi, r_0, r_{\pi/2}\} = \langle R_\pi, r_0\rangle$, while for the second $J = \mathsf{D}_4$.

Qu 3.7: Let S be the subset of $\mathbb{R}^2$ consisting of points $(3n+1, 4m-2)$ (with $m, n \in \mathbb{Z}$). Find the set (group) of translations of $\mathbb{R}^2$ preserving the set S; that is, find

$$L = \left\{\mathbf{v} \in \mathbb{R}^2 \mid \mathbf{x}+\mathbf{v} \in S \ \ \forall \mathbf{x} \in S\right\}.$$

[Hint: Let $\mathbf{t_v}(\mathbf{x}) = \mathbf{y}$. Then $\mathbf{y} = \mathbf{x}+\mathbf{v} \in S$, or $\mathbf{v} = \mathbf{y}-\mathbf{x}$ (with $\mathbf{x}, \mathbf{y} \in S$).]

Solution Let $\mathbf{x}, \mathbf{y} \in S$. Then there are integers m, m', n, n' such that

$$\mathbf{x} = (3n+1, 4m-2), \qquad \mathbf{y} = (3n'+1, 4m'-2).$$

Then in order for $\mathbf{t_v}(\mathbf{x}) = \mathbf{y}$ we require $\mathbf{v} = \mathbf{y}-\mathbf{x} = (3(n'-n), 4(m'-m))$. Since n, n', m, m' are arbitrary integers, it follows that

$$L = \{(3p, 4q) \mid p, q \in \mathbb{Z}\}.$$

In other words, $L = 3\mathbb{Z} \times 4\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$. Note this is a subgroup of $\mathbb{R}^2$ (also of $\mathbb{Z}^2$).

Qu 3.8: [Adapted from past exam] Consider the following subset of the plane:

$$S = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in 2\mathbb{Z},\ x + \tfrac{1}{2}y \in 2\mathbb{Z}\right\}.$$

We wish to show first this is a lattice.

(i). Define the notion of a lattice in the plane.

(ii). Show that the two vectors $\mathbf{u}_1 = (2,0)$ and $\mathbf{u}_2 = (0,4)$ both belong to S, and sketch a diagram showing all the points of S that lie in the rectangle $0 \le x \le 6$ and $0 \le y \le 8$. Deduce that $S \neq \mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(iii). Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $S = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case, and hence deduce that S is a lattice.

(iv). Show that the point group of the lattice S has order 4 by finding appropriate elements of its symmetry group $\mathcal{W}_S < \mathsf{E}(2)$, expressed in the form $(A \mid \mathbf{v})$.

(v). Define a glide reflection. Find a glide reflection $(A \mid \mathbf{v}) \in \mathcal{W}_S$ whose line of reflection is not the line of reflection of a reflection symmetry of S.

Solution Left to you ...

Qu 3.9: Let $u > 0$ and consider the 1-dimensional lattice $\mathbb{Z}\{u\}$. Show that the infinite dihedral group $\mathsf{Dih}(\infty)$ (see appendix) acts on this lattice, via

$$a \cdot x = -x, \quad \text{and} \quad b \cdot x = u - x.$$

(You need to show that these two transformations do indeed preserve L and that they satisfy the relations defining $\mathsf{Dih}(\infty)$).)

Solution The infinite dihedral group is $\mathsf{Dih}(\infty) = \langle a, b \mid a^2 = b^2 = e \rangle$ (see the appendix). For this action, all we need check are the following: $a \cdot (a \cdot x) = -(-x) = x = a^2 \cdot x$ and $b \cdot (b \cdot x) = u - (u - x) = x = b^2 \cdot x$, since $a^2 = b^2 = e$. [Question: how does the element $(ab) \in \mathsf{Dih}(\infty)$ act? (It has infinite order in the group.)]

Qu 3.10: Consider the planar lattice $L = \mathbb{Z}\left\{\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$. Show this is an oblique lattice, and by choosing two appropriate elements, show that it contains a subset which is a rectangular lattice.

Solution Begin by drawing several points of the lattice. It is clear that the smallest non-zero element of the lattice is $(1,2)$. So let $\mathbf{a} := (1,2)^T$, and then $|\mathbf{a}| = \sqrt{5}$. The next shortest is $(-2,2)^T$ or $(2,-2)^T$: the one for which $|\mathbf{a}-\mathbf{b}| < |\mathbf{a}+\mathbf{b}|$ is $\mathbf{b} = (-2,2)^T$, where $|\mathbf{a}-\mathbf{b}| = 3$ and $|\mathbf{a}+\mathbf{b}| = \sqrt{17} > 3$. Thus

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{a}-\mathbf{b}| < |\mathbf{a}+\mathbf{b}|$$

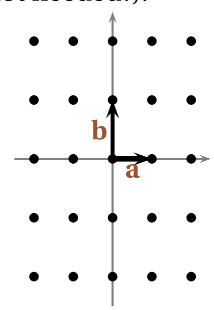
which is the condition for an oblique lattice.
For the rectangular lattice, we can take generators, $\mathbf{a}' = (3,0)^T$ and $\mathbf{b}' = 3\mathbf{a} - \mathbf{a}' = (0,6)^T$. Then $\mathbf{a}', \mathbf{b}'$ generate a rectangular lattice.

Qu 3.11: Which of the 5 types of lattice is $L = \mathbb{Z}\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$?

Solution Centred rectangular!

Qu 3.12: Let $\mathbf{a}, \mathbf{b}$ be two perpendicular vectors of different lengths in $\mathbb{R}^2$, say $|\mathbf{b}| > |\mathbf{a}| > 0$, and let $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ be the resulting rectangular lattice. Let $T \in \mathsf{E}(2)$ be any of the reflections that preserve L. Show that there is $\mathbf{v} \in L$ such that either $T = (r_0 \mid \mathbf{v})$ or $T = (r_{\pi/2} \mid \mathbf{v})$. Deduce that the group $\mathcal{W}_L$ of all symmetries of this lattice is generated by $\{\mathbf{t_a}, \mathbf{t_b}, R_\pi, r_0\}$ (why is $r_{\pi/2}$ not needed?).

Solution



As shown in lectures, if $(A \mid \mathbf{v})$ preserves L then $\mathbf{v} \in L$.

In this case the possibilities for A are I, R_π, r_0 and $r_{\pi/2}$ (they form the point group of the rectangular lattice). The group $\mathcal{W}_L$ is therefore generated by $\{\mathbf{a}, \mathbf{b}, R_\pi, r_0, r_{\pi/2}\}$. However, $r_{\pi/2} = R_\pi r_0$ so is redundant as a generator and every element of $\mathcal{W}_L$ can be written as a composite of elements of $\{\mathbf{a}, \mathbf{b}, R_\pi, r_0\}$.

Qu 3.13: Let $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ be any lattice in the plane. There are many possible centres of symmetry: points $\mathbf{c}$ for which a rotation by π about $\mathbf{c}$ (denoted $R_\pi^{\mathbf{c}}$) is a symmetry of the lattice.

(i). Show that $\mathbf{c}_1 = \frac{1}{2}\mathbf{a}$ and $\mathbf{c}_2 = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ are two such points.

(ii). Show that for each centre $\mathbf{c}$ there is a $\mathbf{v} \in L$ such that $R_\pi^{\mathbf{c}} = (R_\pi \mid \mathbf{v}) \in \mathsf{E}(2)$. If $\mathbf{a}$ and $\mathbf{b}$ are such that there are no reflection symmetries and no other rotations, deduce that the group $\mathcal{W}_L$ of all symmetries of this lattice is generated by $\{\mathbf{a}, \mathbf{b}, R_\pi\}$.

Solution Recall that $(A \mid \mathbf{v})$ denotes the transformation $T(\mathbf{x}) = A\mathbf{x} + \mathbf{v}$. In particular $T(\mathbf{0}) = A\mathbf{0} + \mathbf{v} = \mathbf{v}$; this allows us to easily find $\mathbf{v}$ for any transformation (provided we can do the geometry).
(a) The rotation $R_\pi^{\mathbf{c}}$ by π about $\mathbf{c}$ is, in Seitz notation, $(R_\pi \mid 2\mathbf{c})$ (because $R_\pi^{\mathbf{c}}(\mathbf{0}) = 2\mathbf{c}$). Thus $R_\pi^{\mathbf{c}_1} = (R_\pi \mid \mathbf{a})$. Recall that $R_\pi = -I$. Now suppose $\mathbf{v} \in L$. We want to show $(R_\pi \mid \mathbf{a}) \cdot \mathbf{v} \in L$. There are two ways to proceed.
Firstly, since $\mathbf{v} \in L$ there are integers $m, n \in \mathbb{Z}$ such that $\mathbf{v} = m\mathbf{a} + n\mathbf{b}$. Then

$$(R_\pi \mid \mathbf{a}) \cdot \mathbf{v} = -(m\mathbf{a} + n\mathbf{b}) + \mathbf{a} = (1-m)\mathbf{a} - n\mathbf{b} \in L$$

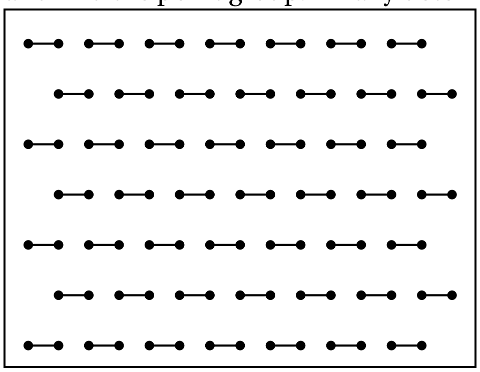
as $(1-m)$ and n are integers. The second way is to note that

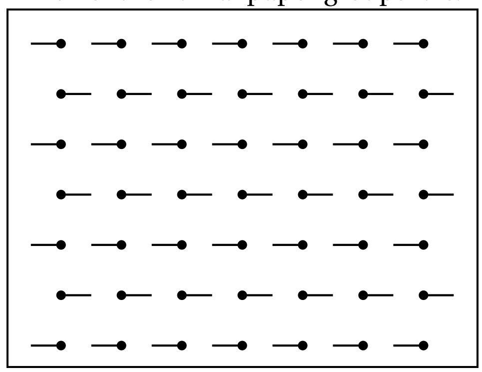
$$(R_\pi \mid \mathbf{a}) \cdot \mathbf{v} = R_\pi \mathbf{v} + \mathbf{a} = \mathbf{a} - \mathbf{v},$$

and since both $\mathbf{a}, \mathbf{v} \in L$ it follows from the fact that L is a group under addition that $\mathbf{a} - \mathbf{v} \in L$.

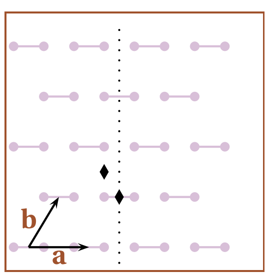
A similar argument holds for $\mathbf{c}_2$.
(b) Let $T = R_\pi^{\mathbf{c}}$. Since $\mathbf{0} \in L$ and T defines a transformation of L, it follows that $\mathbf{v} = T(\mathbf{0}) \in L$. Thus in particular $R_\pi^{\mathbf{c}} = (R_\pi \,|\, \mathbf{v})$ for some $\mathbf{v} \in L$.
Now, any $\mathbf{v} \in L$ can be written $\mathbf{v} = m\mathbf{a} + n\mathbf{b}$ (by definition of L). Thus, any element of $\mathcal{W}_L$ is either a translation $m\mathbf{a} + n\mathbf{b}$ or a rotation $(R_\pi \mid m\mathbf{a} + n\mathbf{b})$. In either case it is a product (= composition) of elements of $\{\mathbf{a}, \mathbf{b}, R_\pi\}$ as required.

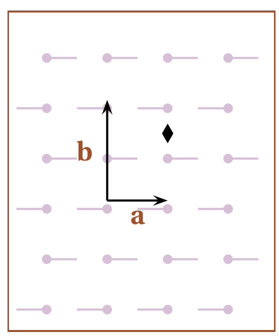
Qu 3.14: For each of the following wallpaper patterns, draw generators of the translation lattice and find the point group. Finally determine which of the 17 wallpaper groups it is.





Solution In the diagrams below, **a** and **b** generate the lattice of translations (the first is centred rectangular and the second is a rectangular lattice).





For the point group J:

- the lozenge shapes are centres of rotation by π (not the only ones), and hence $R_\pi \in J$;
- in both cases there is a reflection in the horizontal axis, and hence $r_0 \in J$;
- in the first pattern there is a reflection in the dotted line, and hence $r_{\pi/2} \in J$,

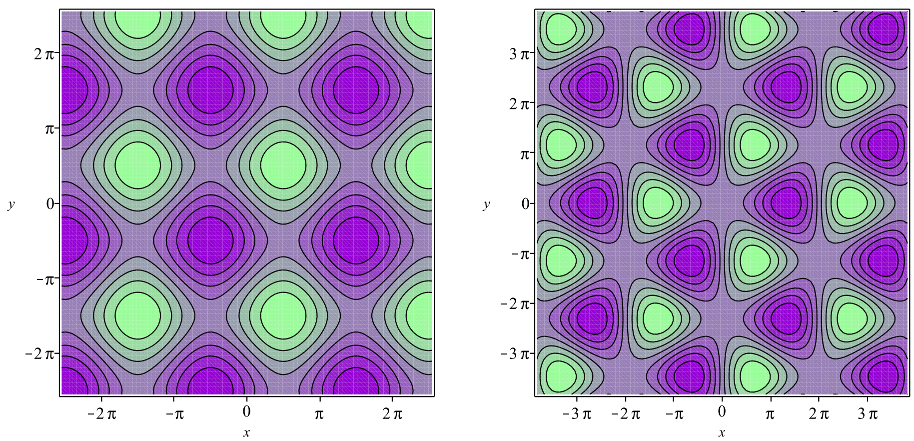


Figure 1: See Problem 3.15. The left-hand figure shows the contours of the function f, the right-hand one the contours of g

- in the second there is a glide reflection given by reflection $r_{\pi/2}$ (in the y-axis) followed by a translation whose glide is $\frac{1}{2}\mathbf{b}$, so again $r_{\pi/2} \in J$;
- there is no centre of rotation with an angle less than π, so that is all the point group.

Hence in both cases, $J = \mathsf{D}_2 = \{I, R_\pi, r_0, r_{\pi/2}\} = \langle R_\pi, r_0\rangle$.
The notation for the two wallpaper groups is cmm and pmg respectively (or 2*22 and 22* in the orbifold notation).

Qu 3.15: Consider the functions of two variables,

$$f(x,y) = \sin(x) + \sin(y) \quad \text{and} \quad g(x,y) \;=\; \sin(x) - 2\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{\sqrt{3}}{2}y\right).$$

The contours of f and g are shown in Figure 1: the lighter, or green, regions are where the function takes positive values and the darker (violet) ones are where the function is negative. Let $\mathcal{W}_f$ and $\mathcal{W}_g$ be their symmetry groups (wallpaper groups).
(a) In each case, find the translation subgroup of $\mathcal{W}$. Which of the 5 types of lattice is this translation subgroup?
(b) Find the point groups J_f and J_g (first just by looking at the diagrams, and then check that these transformations do indeed preserve the function in question).
(c) How is this changed if we allow transformations that change f to $-f$ and g to $-g$? More formally, find the stabilizer of each function under the action of $G = \mathsf{E}(2) \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, -1\}$ and $(T, a)\cdot f = a f \circ T^{-1}$, for $T \in \mathsf{E}(2)$ and $a \in \{\pm 1\}$.

Solution (a) Since sine has period 2π, it follows that $f(x+2n\pi, y+2k\pi) = f(x,y)$. The lattice of translational symmetries of f is therefore generated by $\mathbf{a} = (2\pi, 0)$ and $\mathbf{b} = (0, 2\pi)$. This generates a square lattice, and we can see this in the diagram.
For g we need to look at the diagram. We can see there are rotations by $2\pi/3$ (about some points), so the point group will contain $R_{2\pi/3}$. The only lattice whose point group contains this is the hexagonal lattice, where
$J = \mathrm{D}_6$ (although $J_g = \mathrm{D}_3$ as we check below).
(b) For f the point group is (from the diagram) $J_f = \mathrm{D}_4$ (the symmetry of the square). However, these do not act as rotations etc about the origin. Instead we can rotate about the point $(\frac{\pi}{2}, \frac{\pi}{2})$ (the centre of a green 'square'). The rotation by $\pi/2$ about that point has Setiz symbol $(R_{\pi/2} \mid (0,\pi))$ and this acting on (x,y) gives

$$(x,y) \mapsto R_{\pi/2}(x,y) + (\pi, 0) = (-y+\pi,\, x)$$

Then substituting into f:

$$f(-y+\pi,\, x) = \sin(-y+\pi) + \sin(x) = \sin(y) + \sin(x) = f(x,y).$$

Similarly there are reflections, such as that in the line $y = \pi/2$, which is given by $(x,y) \mapsto (x, \pi - y)$ (with Seitz symbol $(r_0 \mid (0,\pi))$ and a similar argument shows $f(x, \pi - y) = f(x,y)$. These two generate the point group.
For g, the diagram shows there is a rotation by $2\pi/3$ about the origin (also other centres). This is the transformation

$$(x,y) \mapsto \left(-\frac{x}{2} - \frac{\sqrt{3}y}{2},\ \frac{\sqrt{3}x}{2} - \frac{y}{2}\right).$$

Substituting this into f

$$\begin{aligned}
f(R_{2\pi/3}(x,y)) &= -\sin\left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right) + 2\sin\left(\frac{x}{4} - \frac{\sqrt{3}y}{4}\right)\cos\left(\frac{3x}{4} - \frac{\sqrt{3}y}{4}\right) \\
&= -\sin\left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right) + \sin(x) - \sin\left(-\frac{x}{2} + \frac{\sqrt{3}y}{2}\right) \\
&= \sin(x) + \sin\left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right) - \sin\left(\frac{x}{2} - \frac{\sqrt{3}y}{2}\right) \\
&= \sin(x) - 2\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{\sqrt{3}}{2}y\right) \ = \ f(x,y),
\end{aligned}$$

as required. (We used the identity $2\sin(A)\cos(B) = \sin(A+B) - \sin(A-B)$ twice.)
Another generator of the point group is r_0, which just changes the sign of y and that is an obvious symmetry of g. This proves that g is invariant under the point group.
(c) If we allow changes in sign, then for f, the lattice of translations changes, but not the point group (it is still D_4), while for g the point group changes to D_6: a rotation by $2\pi/6 = \pi/3$ about the origin changes the sign of g, as does the reflection $r_{\pi/2}$ (mapping x to $-x$).

Qu 3.16: Refer to Example 3.11, and choose the origin to be at the centre of one of the lozenges. Here we discuss how the group of symmetries is generated. Show that each of the following are in the symmetry group:

$$(R_{\pi/2}\,|\,\mathbf{e}_1), \quad (r_0\,|\,\mathbf{0})$$

where $\mathbf{e}_1 = (1,0)^T$.
(a) Show that the product (composite) $g = (R_{\pi/2}\,|\,\mathbf{e}_1)(r_0\,|\,\mathbf{0})$ is a glide-reflection, and find the line of reflection.
(b) Show that g^2 is one of the vectors that generate the lattice of translations.
(c) Show the other generator is the square of the 'reverse' product $k = (r_0\,|\,\mathbf{0})(R_{\pi/2}\,|\,\mathbf{e}_1)$.
(d) Conclude that the wallpaper group for this pattern is generated by $(R_{\pi/2}\,|\,\mathbf{e}_1)$ and $(r_0\,|\,\mathbf{0})$

Solution First from the diagram it is clear that $(r_0\,|\,\mathbf{0})$ (reflection in the x-axis) is in the symmetry group. The other element $(R_{\pi/2}\,|\,\mathbf{e}_1)$ represents a rotation by $\pi/2$ about the point $(\frac{1}{2},\frac{1}{2})$ (in the centre of a square, like the red dot). And this is also a symmetry as discussed in the example.
(a) From the formula for composing Seitz symbols, $g = (r_{\pi/4}\,|\,\mathbf{e}_1)$. This is a glide reflection, and writing $\mathbf{e}_1$ as $\mathbf{v}_\perp + \mathbf{v}_\parallel$ we see $\mathbf{v}_\perp = (\frac{1}{2},-\frac{1}{2})^T$ and $\mathbf{v}_\parallel = (\frac{1}{2},\frac{1}{2})^T$. Thus the line of reflection is the line with angle $\pi/4$ with the positive x-axis, and translated by $\frac{1}{2}\mathbf{v}_\perp$. To be more specific, we could give its Cartesian equation: it has slope 1, so it must have equation $y = x + c$. Since it passes through the point $\frac{1}{2}\mathbf{v}_\perp = (\frac{1}{4},-\frac{1}{4})$, one finds $c = -\frac{1}{2}$.
(b) $g^2 = (r_{\pi/4}\,|\,\mathbf{e}_1)^2 = (I\,|\,\mathbf{e}_1 + r_{\pi/4}\mathbf{e}_1) = (I\,|\,\mathbf{e}_1 + \mathbf{e}_2)$, and this is one of the two translation vectors shown in the example.
(c) Now $k = (r_0\,|\,\mathbf{0})(R_{\pi/2}\,|\,\mathbf{e}_1) = (r_{-\pi/4}\,|\,\mathbf{e}_1)$. Thus $k^2 = (I\,|\,\mathbf{e}_1 - \mathbf{e}_2)$ and this is (the negative of) the other translation vector.
(d) It follows that all the symmetries of the pattern can be written in terms of $(R_{\pi/2}\,|\,\mathbf{e}_1)$ and $(r_0\,|\,\mathbf{0})$.

Qu 3.17: If we identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$, then there are some famous lattices using the complex numbers:

- Gaussian integers $G = \{a + b\mathrm{i} \mid a, b, \in \mathbb{Z}\} = \mathbb{Z}\{1, \mathrm{i}\}$.
- Eisenstein integers: $E = \{a + b\omega \mid a, b, \in \mathbb{Z}\} = \mathbb{Z}\{1, \omega\}$, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3} = \frac{1}{2}(-1 + \sqrt{3}\,\mathrm{i})$.

Determine which of the 5 types each of these lattices is. [The interesting thing about these lattices is that they are not only groups, but rings as well, as you can check. For the Eisenstein case, one uses the fact that $\omega^2 + \omega + 1 = 0$.]

Solution The Gaussian integers form a square lattice, while the Eisenstein integers form a triangular lattice.

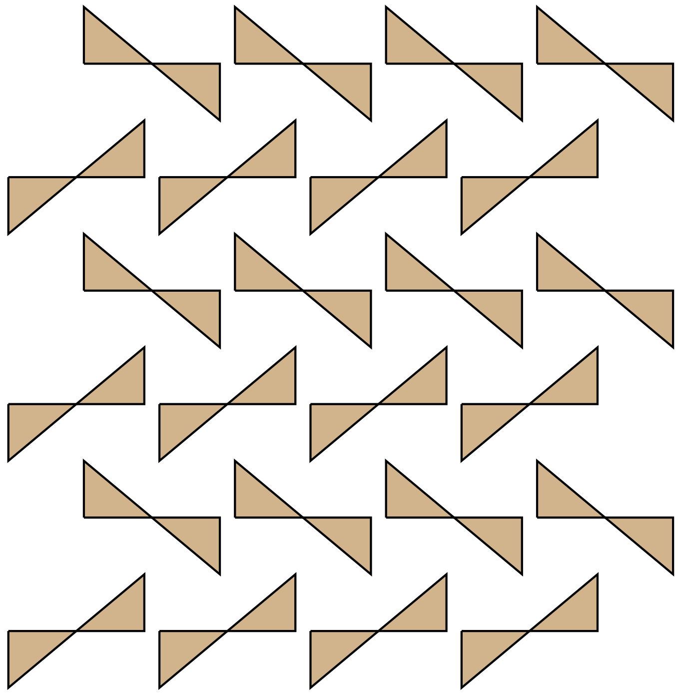


Figure 2: A pattern with symmetry group 22× or pgg, see Problem 3.18

Qu 3.18: Here we look at the wallpaper group pgg, or 22×, see Fig. 2. In the figure, there are two types of 'widget': one whose diagonal edge has positive slope and one with negative slope. Call these positive and negative widgets, respectively.

(i). Show on the diagram generators of the lattice of translations, which is a rectangular lattice. Why is it not a centred rectangular lattice?

(ii). Find all centres of rotation (by π).

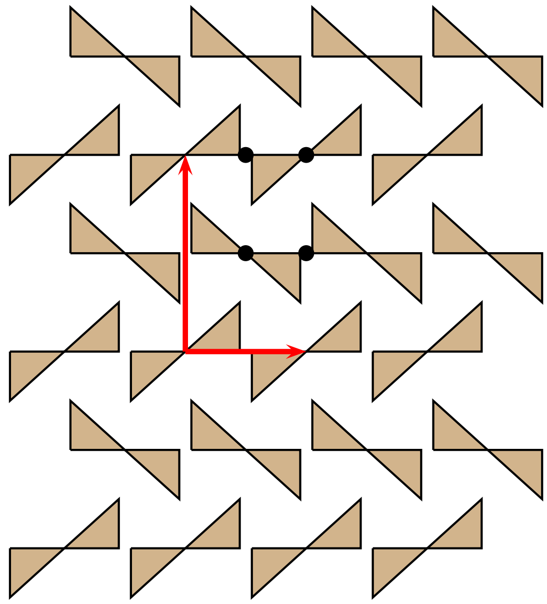
(iii). There are no reflection symmetries of this pattern, but there are both horizontal and vertical glide reflections. Let $\mathbf{a},\mathbf{b}$ be a shortest pair of vectors for the translation lattice, and let $\mathbf{u} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Both by drawing diagrams and by calculation, show that, after suitably choosing an origin, the glide reflections

$$T_1 = (r_0 \,|\, \mathbf{u}), \quad \text{and} \quad T_2 = (r_{\pi/2} \,|\, \mathbf{u})$$

belong to the symmetry group of this pattern.

By considering the squares of these glide-reflections and their products, show that the wallpaper group 22× (or pgg) is generated by T_1 and T_2 (cf. Table 3.2).

Solution See adjacent diagram.



(i) The two red vectors generate the lattice of translations. Call these $\mathbf{a}$ and $\mathbf{b}$. The vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ that would be required for a centred rectangular lattice is not a symmetry because it maps positive widgets to negative widgets, and vice versa.
(ii) Four centres of rotation are shown as large dots, any other centre is a translation of one of these four by the lattice of translations.
(iii) Both r_0 and $r_{\pi/2}$ change positive widgets into negative widgets and *vice versa*. Shifting by $\mathbf{u}$ then restores the type of widget.
One of their products is $(r_0 \,|\, \mathbf{u})(r_{\pi/2} \,|\, \mathbf{u}) = (R_\pi \,|\, \mathbf{u} + r_0\mathbf{u}) = (R_\pi \,|\, \mathbf{e}_1)$. Therefore,

$$(I \,|\, -\mathbf{e}_1)(r_0 \,|\, \mathbf{u})(r_{\pi/2} \,|\, \mathbf{u}) = (R_\pi \,|\, \mathbf{0})$$

as required.
finish this

Qu 3.19: Find all homomorphisms
(a) from $\mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and
(b) from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. [Hint: If H is a cyclic group generated by a, and $\phi : H \to G$ a homomorphism, then ϕ is entirely determined by knowing $\phi(a)$, because $\phi(a^2) = \phi(a)^2$ etc.]

Solution Write the groups additively (addition modulo 4 and 6 respectively). The first group $\mathbb{Z}_4$ is generated by 1, so we need only state the value of $\phi(1)$. It is of course an element of $\mathbb{Z}_6$, and we consider case by case (recall that if $\phi : H \to G$ is a HM then $\phi(e_H) = e_G$, or in this case $\phi(0) = 0$). The only 'obstruction' to choosing $\phi(1)$ is that, because $0 = 4$ in $\mathbb{Z}_4$, we need $\phi(0) = \phi(4) = 4\phi(1)$, so we require $4\phi(1) = 0$ in $\mathbb{Z}_6$.

- $\phi(1) = 0$: This is the trivial HM $\phi(a) = 0$ for all $a \in \mathbb{Z}_4$.
- $\phi(1) = 1$: This does not define a HM, since $\phi(4) = 4\phi(1) = 4 \neq 0$ in $\mathbb{Z}_6$.
- $\phi(1) = 2$: $\phi(0) = \phi(4) = 4\phi(1) = 8 \neq 0$, so not a HM.
- $\phi(1) = 3$: $\phi(0) = \phi(4) = 4\phi(1) = 12 = 0$ (in $\mathbb{Z}_6$), so this is a HM.

- $\phi(1) = 4$ and 5: these are also not homomorphism for similar reasons.

There are therefore only two homomorphisms $\mathbb{Z}_4$ to $\mathbb{Z}_6$, one of which is trivial and the other has image $\{0,3\} \subset \mathbb{Z}_6$.

Qu 3.20† (a) Prove the following lemma:
Let G be a group and $H \lhd G$ a normal subgroup. Then the action of G on itself by conjugation restricts to an action of G on H. Moreover, if H is abelian, this defines an action of the quotient group G/H on H.
(b) Let $G = \mathsf{D}_n$ and $H = \mathsf{C}_n$ (which is a normal subgroup). Determine the resulting action of G/H on H.
(c) Deduce Proposition 3.8 from this lemma.

Solution (a) *Proof of Lemma:* Recall that G acts on itself by conjugation $\mu : G \to S(G)$ with $\mu(g)(k) = gkg^{-1}$. We have seen that this is an isomorphism. Therefore, $\mu : G \to \mathrm{Aut}(G)$ where $\mathrm{Aut}(G)$ is the group of isomorphisms of G. The main point is that if $h \in H$, then $ghg^{-1} \in gHg^{-1} = H$ as $H \lhd G$. So this action defines an action of G on H; that is, $\mu(g) : H \to H$, or $\mu(G) \in S(H)$.
Now consider the factor group G/H whose elements are cosets of the form gH. We want to show that $\mu(gH) : H \to H$ is well defined (independent of the choice of representative in the coset gH). (This is false if H is not abelian.) Consider two elements of gH, they are g and gh for some $h \in H$. For $k \in H$, we get

$$\begin{aligned} \mu(gh)(k) &= ghk(gh)^{-1} \\ &= ghkh^{-1}g^{-1} \\ &= ghh^{-1}kg^{-1} \\ &= gkg^{-1} \\ &= \mu(g)(k). \end{aligned}$$

Thus $\mu(gh)$ and $\mu(g)$ have the same effect on H and we are done.
(b) Write $R = R_{2\pi/n} \in \mathsf{C}_n$. Now,

$$\mathsf{C}_n = \langle R \rangle, \quad \text{and} \quad \mathsf{D}_n = \langle R, r_0 \rangle,$$

and $\mathsf{D}_n/\mathsf{C}_n \simeq \mathbb{Z}_2$. We want to know how this $\mathbb{Z}_2$ acts on C_n. Take any representative of the coset $r_0\mathsf{C}_n$, say $g = r_0$ itself. A simple calculation shows (using for example formulae from Chapter 2), for any θ,

$$r_0 R_\theta r_0^{-1} = R_{-\theta}.$$

(Of course, $r_0^{-1} = r_0$.) In particular $r_0 R r_0 = R^{-1}$. So that is the answer: the non-identity element of $\mathsf{D}_n/\mathsf{C}_n$ acts on C_n by $R \mapsto R^{-1}$, and hence by $R^k \mapsto R^{-k}$.

(c) *Proof of Proposition 3.8* Let $\mathcal{W} \subset \mathsf{E}(2)$ with $L := \mathcal{W} \cap \mathbb{R}^2$ (subgroup of translations) and $J := \pi(\mathcal{W}) \subset \mathsf{O}(2)$ the point group. We claim that L is abelian and normal in $\mathcal{W}$ and that J is isomorphic to $\mathcal{W}/L$.

- Since $L \leq \mathbb{R}^2$, L is abelian.
- Recall that $\pi : \mathsf{E}(2) \to \mathsf{O}(2)$ is a homomorphism. Restrict that to $\mathcal{W} \leq \mathsf{E}(2)$.

$$\begin{aligned} \ker(\pi) &= \{(A \mid \mathbf{v}) \in \mathcal{W} \mid A = I\} \\ &= \mathcal{W} \cap \mathbb{R}^2 \\ &= L. \end{aligned}$$

  In particular, $L \lhd \mathcal{W}$ since the kernel of a homomorphism is always a normal subgroup.

- Finally, we use the first isomorphism theorem to conclude that $\mathcal{W}/\ker(\pi) \simeq \mathrm{Im}(\pi)$; that is, $\mathcal{W}/L \simeq J$.

Thus, by the Lemma proved above, with $G = \mathcal{W}$, $H = L$ and so $J = G/H$, we deduce that J acts on the lattice L.

