Solutions for Chapter 2: Euclidean Transformations

Qu 2.1: Deduce from Definition 2.1 that any Euclidean transformation of $\mathbb{R}^n$ is injective.

Solution Let f be a Euclidean transformation. Suppose $\mathbf{x} \neq \mathbf{y}$. Then $|\mathbf{x}-\mathbf{y}| \neq 0$. Since f is an isometry, it follows that $|f(\mathbf{x}) - f(\mathbf{y})| \neq 0$ and therefore $f(\mathbf{x}) \neq f(\mathbf{y})$, which shows f is injective.

Qu 2.2: Show the set $\mathsf{O}(n)$ of orthogonal $n \times n$ matrices forms a subgroup of $\mathsf{GL}(n)$. (Recall (see Appendix) that $\mathsf{GL}(n)$ is the group of all invertible $n \times n$ matrices.)

Solution Since $\mathsf{O}(n)$ is a subset of $\mathsf{GL}(n)$ we need only apply the subgroup criterion.
[**is non-empty:**] $I \in \mathsf{O}(n)$ since $I^T I = I$
[**product+inverse:**] (Here we show both in one go.) Suppose $A, B \in \mathsf{O}(n)$. Then $A^T A = B^T B = I$ (equivalently, $A^{-1} = A^T$ etc). We want to show $AB^{-1} \in \mathsf{O}(n)$. Now,

$$\begin{aligned}
(AB^{-1})^T(AB^{-1}) &= (B^{-1})^T A^T AB^{-1} \\
&= BA^T AB^{-1} \\
&= BIB^{-1} \\
&= BB^{-1} \\
&= I
\end{aligned}$$

as required.

Qu 2.3: Let V be a vector space, which we know is a group under vector addition. For each $\mathbf{v} \in V$ there is the translation $T_\mathbf{v}(\mathbf{x}) = \mathbf{x} + \mathbf{v}$. Show that this defines an action T of V on itself, which coincides with the left translation defined in Section 1.3.

Solution Firstly, for each $\mathbf{v} \in V$, $T_\mathbf{v}$ maps V to itself. We need to show that $\mathbf{v} \mapsto T_\mathbf{v}$ is a homomorphism. That is, we need to show that, for $\mathbf{v}, \mathbf{u} \in V$,

$$T_{\mathbf{v}+\mathbf{u}} = T_\mathbf{v} \circ T_\mathbf{u}.$$

To check this, let $\mathbf{x} \in V$,

$$\begin{aligned}
T_\mathbf{v} \circ T_\mathbf{u}(\mathbf{x}) &= T_\mathbf{v}(\mathbf{x}+\mathbf{u}) \\
&= \mathbf{x}+\mathbf{u}+\mathbf{v} \\
&= \mathbf{x}+\mathbf{v}+\mathbf{u} \\
&= T_{\mathbf{v}+\mathbf{u}}(\mathbf{x}).
\end{aligned}$$

This holds for every $\mathbf{x} \in V$ and hence $T_{\mathbf{v}+\mathbf{u}} = T_\mathbf{v} \circ T_\mathbf{u}$ as required.

Why is this the left translation of V defined in Section 1.3? That action is (in general) given by,

$$\lambda(g)(g') = gg'.$$

In the case where $G = V$, the binary operation is vector addition, so gg' becomes $\mathbf{v}+\mathbf{v}'$. Thus $\lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{v}+\mathbf{x} = \mathbf{x}+\mathbf{v}$, and so $\lambda(\mathbf{v}) = T_{\mathbf{v}}$. (This example explains why the action λ is called (left) translation.)

Qu 2.4: Find the eigenvalues of R_θ and r_α. How are the eigenvectors of r_α related to the line of reflection?

Solution The eigenvalues of R_θ are $\mathrm{e}^{\pm i\theta}$. The eigenvalues of r_α are ± 1.
For r_α, let $\mathbf{v}$ be a vector parallel to the line of reflection. Then $r_\alpha \mathbf{v} = \mathbf{v}$, so $\mathbf{v}$ is an eigenvector with eigenvalue 1. Moreover $r_\alpha \mathbf{v}^\perp = -\mathbf{v}^\perp$, so $\mathbf{v}^\perp$ is an eigenvector with eigenvalue -1.
[Note: for $\theta \neq n\pi$, the eigenvectors of R_θ are complex vectors.]

Qu 2.5: Verify the identities in Eq. (2.8). Let r_α be a reflection, and find the angle of rotation of $r_\alpha R_\theta r_\alpha^{-1}$. Deduce that, for each n, C_n is a normal subgroup of $\mathsf{O}(2)$.

Solution The verifications are left to you. Now $r_\alpha R_\theta = r_{\alpha-\theta/2}$ and so

$$r_\alpha R_\theta (r_\alpha)^{-1} = r_\alpha R_\theta r_\alpha = r_{\alpha-\theta/2} r_\alpha = R_{2(\alpha-\theta/2-\alpha)} = R_{-\theta}.$$

(Notice that this is independent of α.). If $\theta = 2\pi j/n$ then $R_\theta \in \mathsf{C}_n$ and $R_{-\theta} \in \mathsf{C}_n$. Moreover (as we have already observed) $R_\phi R_\theta R_\phi^{-1} = R_\theta$. Thus $r\mathsf{C}_n r^{-1} = \mathsf{C}_n$ and $R\mathsf{C}_n R^{-1} = \mathsf{C}_n$ for all reflections r and rotations R and hence $\mathsf{C}_n \lhd \mathsf{O}(2)$.

Qu 2.6: Show that the map $p : \mathsf{E}(n) \to \mathsf{O}(n)$ given by $p(A \mid \mathbf{v}) = A$ is a homomorphism, and deduce that the set of translations in $\mathsf{E}(n)$ is a normal subgroup.

Solution To be a homomorphism, we need $p((A \mid \mathbf{u})(B \mid \mathbf{v})) = p(A \mid \mathbf{u})p(B \mid \mathbf{v})$. Now,

$$\begin{aligned} p((A \mid \mathbf{u})(B \mid \mathbf{v})) &= p(AB \mid \mathbf{u}+A\mathbf{v}) \\ &= AB. \end{aligned}$$

On the other hand, $p(A \mid \mathbf{u}) = A$ and $p(B \mid \mathbf{v}) = B$, hence $p(A \mid \mathbf{u})p(B \mid \mathbf{v}) = AB$. Therefore p is a homomorphism.
For the second part, we just note that the kernel of this homomorphism is the set of $(A \mid \mathbf{v})$ with $A = I$, which is precisely the set of translations.

Qu 2.7: In contrast to problem 2.6, show that the map $p' : \mathsf{E}(n) \to \mathbb{R}^n$ defined by $p'(A\,|\,\mathbf{v}) = \mathbf{v}$ is *not* a homomorphism.

Solution

$$\begin{aligned} p'((A\,|\,\mathbf{u})(B\,|\,\mathbf{v})) &= p'(AB\,|\,\mathbf{u}+A\mathbf{v}) \\ &= \mathbf{u}+A\mathbf{v}. \end{aligned}$$

On the other hand, $p'(A\,|\,\mathbf{u}) + p'(B\,|\,\mathbf{v}) = \mathbf{u}+\mathbf{v}$ so these are not equal in general and it is not a homomorphism.

Qu 2.8: Describe the transformation of the plane represented by each of the following Seitz symbols:

$$\text{(i) } (I\,|\,\mathbf{v}), \quad \text{(ii) } (R_\pi\,|\,\mathbf{v}), \quad \text{(iii) } (r_{\pi/4}\,|\,\mathbf{v}), \quad \text{(iv) } (r_0\,|\,\mathbf{v}).$$

where $\mathbf{v} = (1,\,1)^T \in \mathbb{R}^2$.

Solution (i) This is simply translation by $\mathbf{v}$, so is the map $(x, y) \mapsto (x+1, y+1)$.
(ii) $(R_\pi\,|\,\mathbf{v})$ is rotation by π about the point $(\frac{1}{2}, \frac{1}{2})$.
(iii) The vector $\mathbf{v}$ is parallel to the line of reflection of $r_{\pi/4}$, so $(r_{\pi/4}\,|\,\mathbf{v})$ is a glide-reflection in that line.
(iv) In this case, the vector $\mathbf{v}$ is neither parallel nor perpendicular to the line of reflection of r_0 (the line $y = 0$, or the x-axis). We therefore need to decompose it into its two parts:

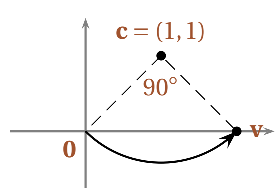
$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}.$$

The transformation $(r_0\,|\,\mathbf{v})$ is therefore a reflection in the line $y = \frac{1}{2}$ followed by a translation by 1 unit to the right.

Qu 2.9: Write the Seitz symbol for each of the following Euclidean transformations of the plane:
(i) the rotation through $\pi/2$ about the point (1,1);
(ii) the reflection in the line $y = x+1$;
(iii) the glide reflection consisting of the reflection in (ii) followed by a translation by (1,1) (which is parallel to the line of reflection).

Solution

(i) The orthogonal part is $R_{\pi/2}$ and $\mathbf{v} = (2,0)$ (see diagram). Seitz symbol is therefore $(R_{\pi/2}\,|\,(2,0)^T)$.
(ii) The orthogonal part is $r_{\pi/4}$. The image of $\mathbf{0}$ under the reflection is $(-1,1)$ and therefore the Seitz symbol is $(r_{\pi/4}\,|\,(-1,1)^T)$.



(iii) Here we can either work it out from first principles (like the two above), or use the Seitz symbol from (ii). The transformation is the composite of the reflection from (ii) followed by a translation, so we have

$$(A \mid \mathbf{v}) = (I \mid (1,1)^T)\,(r_{\pi/4} \mid (-1,1)^T) = (r_{\pi/4} \mid (0,2)^T),$$

using the formula for the product of Seitz symbols.

Qu 2.10: For $\mathbf{v} \in \mathbb{R}^n$, as usual let $T_\mathbf{v}$ denote the translation $\mathbf{x} \mapsto \mathbf{x}+\mathbf{v}$. Let $A \in \mathsf{O}(n)$. Show that conjugation of a translation by A is another translation, and in particular,

$$AT_\mathbf{v}A^{-1} = T_{A\mathbf{v}}.$$

Solution Calculate the effect of $AT_\mathbf{v}A^{-1}$ on a vector $\mathbf{x}$:

$$\begin{aligned} AT_\mathbf{v}A^{-1}(\mathbf{x}) &= AT_\mathbf{v}(A^{-1}\mathbf{x}) \\ &= A(A^{-1}\mathbf{x}+\mathbf{v}) \\ &= AA^{-1}\mathbf{x}+A\mathbf{v} \\ &= \mathbf{x}+A\mathbf{v} \end{aligned}$$

as required, since $T_{A\mathbf{v}}(\mathbf{x}) = \mathbf{x}+A\mathbf{v}$.

Qu 2.11: Use the Seitz symbol to describe the group $\mathsf{E}(1)$ of Euclidean transformations of the line. [First describe its elements, and then the group structure.]

Solution The Seitz symbol of an element of $\mathsf{E}(1)$ is (as in general) $(A \mid u)$, with $A \in \mathsf{O}(1)$ and $u \in \mathbb{R}$. Now A is a 1×1 orthogonal matrix, which just means $A^TA = A^2 = 1$, or $A = \pm 1$. The elements of $\mathsf{E}(1)$ are therefore $(\pm 1 \mid u)$ with $u \in \mathbb{R}$. This element acts on $\mathbb{R}$ by

$$(\pm 1 \mid u)x = \pm x + u.$$

And by the general formula for composition of Seitz symbols, $(A \mid u)(B \mid v) = (AB \mid u + Av)$, with $A, B = \pm 1$.

Qu 2.12: Given any transformation $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, define the 3×3 invertible matrix, written in block form,

$$\psi((A \mid \mathbf{v})) = \left(\begin{array}{c|c} A & \mathbf{v} \\ \hline 0 & 1 \end{array}\right) \in \mathsf{GL}_3(\mathbb{R}).$$

(i) Let $g = (r_{\pi/4} \mid \mathbf{u}) \in \mathsf{E}(2)$, where $\mathbf{u} = (1,1)^T$. Write down $\psi(g)$ and calculate $\psi(g)^2$ and compare with $\psi(g^2)$.
(ii) Show that the map $\psi : \mathsf{E}(2) \to \mathsf{GL}_3(\mathbb{R})$ is a homomorphism.

Solution (a) Now $r_{\pi/4} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and hence

$$\psi(g) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and therefore} \quad \psi(g)^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

We need to compare this with $\psi(g^2)$. Now $g^2 = (r_{\pi/4} \mid \mathbf{u})(r_{\pi/4} \mid \mathbf{u}) = (r_{\pi/4}^2 \mid \mathbf{u} + r_{\pi/4}\mathbf{u}) = (I \mid 2\mathbf{u})$. Thus indeed $\psi(g^2) = \psi(g)^2$.

(b) To show a map ψ is a homomorphism, we only need show that $\psi(gh) = \psi(g)\psi(h)$ (for all g, h) (see the Appendix). Let $g = (A \mid \mathbf{v})$ and $h = (B \mid \mathbf{u})$, then $gh = (AB \mid A\mathbf{u} + \mathbf{v})$. Now,

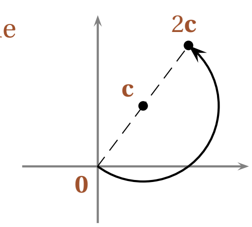
$$\psi(g)\psi(h) = \left(\begin{array}{c|c} A & \mathbf{v} \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} B & \mathbf{u} \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} AB & A\mathbf{u} + \mathbf{v} \\ \hline 0 & 1 \end{array}\right)$$

and this is precisely $\psi(gh)$.

Qu 2.13: (i) A rotation in the plane through an angle π is called a ***half-turn***. By using the homomorphism $\mathsf{E}(2) \to \mathsf{O}(2)$ taking $(A \mid \mathbf{v})$ to A, show that the composite of two half-turns is a translation.
(ii) For $\mathbf{c} \in \mathbb{R}^2$, denote by $h(\mathbf{c})$ the half-turn with centre $\mathbf{c}$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Express the translation $h(\mathbf{b})h(\mathbf{a})$ explicitly in terms of $\mathbf{a}$ and $\mathbf{b}$.

Solution The map $\pi : \mathsf{E}(2) \to \mathsf{O}(2)$ taking $(A \mid \mathbf{v})$ to A is a homomorphism, whose kernel is the subgroup of translations. Let $(R_\pi \mid \mathbf{u})$ and $(R_\pi \mid \mathbf{v})$ be two half turns. Then $\pi((R_\pi \mid \mathbf{u})(R_\pi \mid \mathbf{v})) = R_\pi^2 = I$ so that the product lies in the kernel and is hence a translation.

The first question is how to write $h(\mathbf{a})$ in the form $(A \mid \mathbf{v})$. Firstly $A = R_\pi$ as it's a rotation through π. Secondly, $\mathbf{v}$ is the image of $\mathbf{0}$. Now the half-turn with centre $\mathbf{c}$ maps $\mathbf{0}$ to $2\mathbf{c}$ (see figure). Thus, $\mathbf{v} = 2\mathbf{c}$, and the Seitz symbol of $h(\mathbf{c})$ is $(R_\pi \mid 2\mathbf{c})$.



Therefore, $h(\mathbf{a}) = (R_\pi \mid 2\mathbf{a})$ and $h(\mathbf{b}) = (R_\pi \mid 2\mathbf{b})$. Therefore, using the multiplication rule for the Seitz symbol, which is

$$(A \mid \mathbf{u})(B \mid \mathbf{v}) = (AB \mid \mathbf{u} + A\mathbf{v}),$$

we get

$$h(\mathbf{b})h(\mathbf{a}) = (R_\pi \mid 2\mathbf{b})(R_\pi \mid 2\mathbf{a}) = (R_\pi^2 \mid 2\mathbf{b} + R_\pi(2\mathbf{a})) = (I \mid 2\mathbf{b} - 2\mathbf{a}).$$

Thus $h(\mathbf{b})h(\mathbf{a})$ is the translation by the vector $2(\mathbf{b} - \mathbf{a})$.

Qu 2.14: Complete the following table, showing the geometric type (point/line ...) of the set of points fixed under each type of Euclidean transformation:

Solution

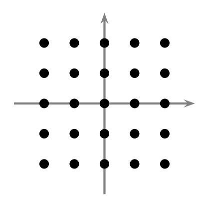
Type	fixed point set
Identity	$\mathbb{R}^2$
Translation	empty set
Rotation	point
Reflection	line
Glide-reflection	empty set

These are all different, except that both translations and glide-reflections have no fixed point. (You might like to show that the square of a glide reflection is a translation.)

Qu 2.15: If r_L is the reflection in the line L, and $g \in \mathsf{E}(2)$ is a Euclidean transformation, show that $g\,r_L\,g^{-1}$ is the reflection in the line $g(L)$. [Hint: use Problem 2.14.]

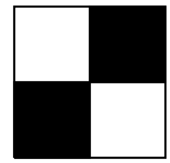
Solution There are several ways to do this; this is the most geometric:
A reflection is the only type of transformation that fixes every point on a line, in this case on L (see Question 2). So we need to show that gr_Lg^{-1} fixes every point on $g(L)$. Let $x \in g(L)$. Then $y = g^{-1}(x) \in L$. Then $r_L(y) = y$ and finally, since $g(y) = x$, we get $gr_Lg^{-1}(x) = x$. That is, gr_Lg^{-1} fixes all the points in $g(L)$. The only other transformation which fixes all the points of $g(L)$ is the identity, which gr_Lg^{-1} clearly is not, so therefore gr_Lg^{-1} is the reflection in $g(L)$.
An alternative method is to use the Seitz symbol $(r\,|\,\mathbf{v})$ and then conjugate with another Seitz symbol $g = (A\,|\,\mathbf{u})$.

Qu 2.16: Consider the subset $L \subset \mathbb{R}^2$ shown to the right consisting of all points in the plane both of whose coordinates are integers (that is, $L = \mathbb{Z}^2$). Find the rotations and reflections in $\mathsf{O}(2)$ which preserve L (that is, map points in L to points in L).



Solution Rotations through $\pi/2, \pi$, and $3\pi/2$, and reflections in the lines $y = 0$, $x = 0$, $y = x$, $y = -x$. (That is $r_0, r_{\pi/4}$, $r_{\pi/2}$ and $r_{3\pi/4}$.) This is the group D_4.

Qu 2.17: Consider a 2×2 chessboard with 2 black squares and 2 white as shown.



(a) Which reflections and rotations preserve the chessboard with its colouring: i.e. sending black squares to black and white to white. You should use the notation introduced on the sheet on symmetries of the square.
(b) How do we know in advance (i.e., without finding them) that these transformations will form a subgroup of D_4?
[Hint for (b): Consider the action of D_4 on the set $\{C, C'\}$ where C is the chessboard shown, and C' the same board but with the colours swapped.]

Solution (a) For example, rotation by $\pi/2$ reverses the colours, as does the reflection r_0. The subset (subgroup) of elements preserving the colours is

$$H = \{e, R_\pi, r_{\pi/4}, r_{3\pi/4}\}.$$

(b) Why should this be a subgroup? There are two ways to answer this. Firstly, the calculating way: using the subgroup criterion by calculating with the 4 elements. Alternatively, there is the more 'conceptual' approach as follows. Consider the set $\{C, C'\}$ as described. Then each element of D_4 either preserves the colours, so sends C to itself, or swaps the colours, sending C to C'; that is, D_4 acts on this set $\{C, C'\}$. The elements of H are those that fix C: that is, H is the stabilizer of C. And we have proved that stabilizers are subgroups.

Qu 2.18: Continuing the previous question, now consider the action of $\mathbb{Z}_2 = \{e, c\}$ (with $c^2 = e$) on the chessboard where $c \in \mathbb{Z}_2$ acts by changing the colour of every square: black to white and white to black. Combine this with the action of D_4 by rotations and reflections to form an action of the product $G = \mathsf{D}_4 \times \mathbb{Z}_2$. For example, $(R_{\pi/2}, c)$ acts by rotating by $\pi/2$ and then changing the colours. What is the symmetry group of the chessboard, as a subgroup of G? (In other words, which elements of G preserve the chessboard with its colouring?)

Solution We could write out a list of all 8 elements, such as $(R_{\pi/2}, c)$ and (R_π, e), but here is another way of expressing the answer. Each element $g \in \mathsf{D}_4$ either preserves or reverses the colouring. Define a homomorphism $\varphi : \mathsf{D}_4 \to \mathbb{Z}_2$ by

$$\varphi(g) = \begin{cases} e & \text{if } g \text{ preserves colour} \\ c & \text{if } g \text{ reverses colour.} \end{cases}$$

(Why is it a homomorphism?) The subgroup of $\mathsf{D}_4 \times \mathbb{Z}_2$ that preserves the colouring is

$$\{(g, \varphi(g)) \mid g \in \mathsf{D}_4\}.$$

(This is the graph of φ.) This is because whenever g reverses the colour, so $\varphi(g)$ reverses it back again.

Qu 2.19: Find all homomorphisms from $\mathbb{Z}_2$ to each of $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

Solution Write $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition modulo n. In each case it is enough to say what $\phi(1)$ is. The answers are:

- For $\phi : \mathbb{Z}_2 \to \mathbb{Z}_3$, the only possibility is the trivial homomorphism $\phi(1) = 0$.
- For $\phi : \mathbb{Z}_2 \to \mathbb{Z}_4$ there are two possibilities: the trivial one $\phi(1) = 0$ and $\phi(1) = 2 \pmod 4$. If you try $\phi(1) = 1$ or $= 3$ then $\phi(0) = \phi(1+1) = \phi(1) + \phi(1) \neq 0$ giving a contradiction.

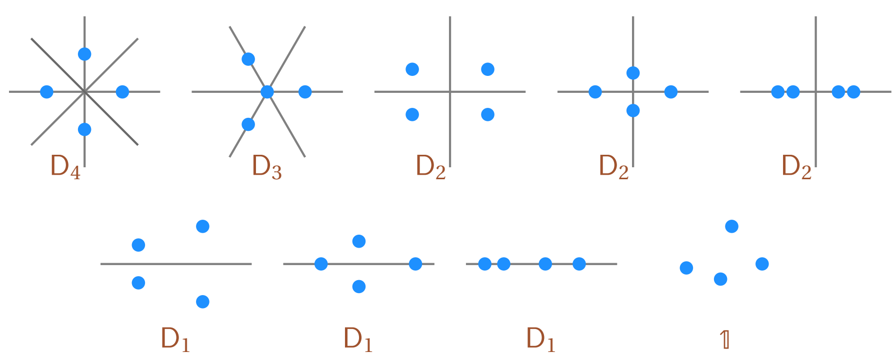
Qu 2.20[†] Describe all rotations (centre + angle) and lines of reflection in the plane which preserve the set $\mathbb{Z}^2$ described in Problem 2.16.

Solution (Not written yet)

Qu 2.21[†] Find the possible symmetry types of sets of 4 distinct points in the plane.

Solution The possibilities are,

Stabilizer	geometry
D_4	at the vertices of a square
D_3	3 at the vertices of an equilateral triangle, and one in the centre
D_2	vertices of a rectangle
D_2	vertices of a rhombus
D_2	collinear points, symmetrically placed
D_1	symmetric trapezium (no points fixed by the reflection)
D_1	kite (two points fixed by reflection)
D_1	collinear points, at general positions (all points fixed by reflection)
$\mathbb{1}$	4 points at general positions



The lines shown are the lines of reflection in D_n.

Qu 2.22† Find the possible symmetry types of *quadrilaterals* in the plane. (A quadrilateral consists of 4 points *in order*: that is, in $ABCD$ there is no edge joining A to C nor B to D, and as a result the answer should be different to the previous question.)

Solution All of the diagrams above can be made into a symmetric quadrilateral except the one with D_3 symmetry.

