Problems for Chapter 6: Periodic Motion

Qu 6.1: Use the property of uniqueness of solutions of ODEs to show that if γ is a solution for which there is a $T > 0$ such that $\gamma(T) = \gamma(0)$ then $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$.

Qu 6.2: Consider a system of 4 identical coupled cells with symmetry D_4. Draw the cell diagram for such a system. Let $\gamma(t)$ be a periodic orbit with symmetry $\widetilde{\mathbb{Z}_4}$ generated by $((1\,2\,3\,4), \frac{1}{4}) \in S_4 \times S^1$ and period T. State the relation between the cells after a quarter of a period. If $x_1(t) = \sin(t)$ (with $T = 2\pi$), deduce the form of $x_j(t)$ for $j = 2,3,4$. Plot the graphs of the 4 functions $x_j(t)$, on the same diagram.

Qu 6.3: Consider a system of 3 identical coupled cells with symmetry S_3. Draw the cell diagram for such a system. Let $\gamma(t)$ be a periodic orbit with symmetry $\widetilde{\mathbb{Z}_2}$ generated by $((1\,2), \frac{1}{2}) \in S_3 \times S^1$ and period T. State the relation between the 3 cells after half a period, and deduce the period of cell 3.

Qu 6.4: By considering the subgroups of $G = \mathsf{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and all possible homomorphisms from these to $S^1 = \mathbb{R}/\mathbb{Z}$, find all possible symmetry groups of periodic orbits in a system with D_2 symmetry. [Hint: there are 5 subgroups, and these have 1, 2, 2, 2 and 4 homomorphisms respectively giving 11 possible symmetry groups in all.]

Qu 6.5: Repeat Example 6.10, but for the D_4 action on $\mathbb{R}^2$, showing the existence of periodic orbits with symmetries shown in Figure 6.1(b).

Qu 6.6: Find all 27 complex-axial symmetry groups of the action of $\mathbb{T}_d$ on $\mathbb{R}^3$ described in Section 4.5.

Qu 6.7: Suppose G acts on $\mathbb{R}^n$ and let X denote the space of all continuous maps $\gamma : S^1 \to \mathbb{R}^n$. There is an action of $G \times S^1$ on this space: if $\gamma \in X$ then define $(g,\theta)\cdot\gamma$ to be the map,

$$\big((g,\theta)\cdot\gamma\big)(t) = g\cdot\big(\gamma(t-\theta)\big).$$

Verify that this defines an action, and show that the stabilizer of an element γ is precisely its symmetry group Σ_γ.

