Problems for Chapter 5: ODEs

Qu 5.1: Let $\mathbb{Z}_2 = \langle r\rangle$ act on $\mathbb{R}$ by $r \cdot x = -x$. Show that the differential equation $\dot{x} = \sin(2x)$ has symmetry group $\mathbb{Z}_2$. Let $x(t)$ be the solution with initial value $x(0) = 1$, and let $u(t)$ be the solution with initial value $u(0) = -1$. How are $x(t)$ and $u(t)$ related?

Qu 5.2: Let D_3 be the usual dihedral subgroup of order 6 of $\mathsf{O}(2)$, generated by r_0 and $R_{2\pi/3}$. Consider the system of ODEs

$$\begin{cases} \dot{x} &= x + x^2 - y^2 \\ \dot{y} &= y - 2xy \end{cases}$$

(a) Show this system has D_3 symmetry.
(b) List all three axial subgroups of D_3. By choosing one of these, find all equilibria of this system with axial symmetry, and explain briefly why it is enough to consider only one of the axial subgroups.

Qu 5.3: Let L be an $n \times n$ matrix, and consider the first order ODE $\dot{\mathbf{x}} = L\mathbf{x}$ on $\mathbb{R}^n$. Show this is equivariant for a linear action of a group G if and only if L commutes with all the matrices in the representation of G.

Qu 5.4: Consider the following family of system of ODEs in the plane

$$\begin{cases} \dot{x} &= ax + x^2 - y^2 \\ \dot{y} &= ay - 2xy \end{cases}$$

Here $a \in \mathbb{R}$ is a parameter. This is similar to a previous question, and the system has D_3 symmetry. Describe the bifurcations of equilibrium points that occur on the lines of symmetry as a is varied through $a = 0$.

Qu 5.5: Consider the similar system with symmetry D_4:

$$\begin{cases} \dot{x} &= x + x^3 - 3xy^2 \\ \dot{y} &= y - 3x^2y + y^3. \end{cases}$$

(a) Show this system has D_4 symmetry.
(b) List all axial subgroups of D_4, and find all equilibria of this system with axial symmetry. (Explain briefly why in this case it is *not* enough to consider only one of the axial subgroups.)

Qu 5.6: Check that the two systems in Examples 5.4 have symmetry S_3 and $\mathbb{Z}_4$ respectively.

Qu 5.7: Consider the following system of ordinary differential equations,

$$\begin{cases} \dot{x} &= -x + yz^2 \\ \dot{y} &= -y + xz^2 \\ \dot{z} &= z(1 + xy - z^2). \end{cases} \qquad (*)$$

Consider the action of the group $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i). Show that the matrices A, B, C do indeed generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (you need to show that $A^2 = I$ etc, and A, B, C all commute).

(ii). Show that the system $(*)$ has symmetry G.

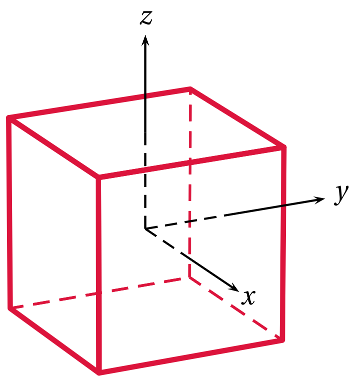
(iii). Deduce that the x-y plane and the z-axis are each invariant under the evolution of the system, stating carefully any results used.

(iv). Can you find other invariant subspaces?

(v). Find all the equilibrium points that lie on these subspaces.

(vi). Find the unique solution to this system with initial value $(x, y, z) = (1, 1, 0)$. What is the limit as $t \to \infty$ of this solution?

Qu 5.8: The octahedral group $\mathbb{O}_h$ is the group of all symmetries of the cube (including reflections). With vertices at the 8 points $(\pm1, \pm1, \pm1)$, it is generated as follows.



Generators:

$$R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_d = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$r_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Here R_z is a rotation about the z-axis by $\pi/2$, R_d is a rotation by $2\pi/3$ about the diagonal $x = y = z$, and r_z is the reflection in the x-y plane.
Consider the following family of potential functions in 3-D:

$$V = \lambda\left(x^2 + y^2 + z^2\right) - 2\left(x^4 + y^4 + z^4\right) + 3\left(x^2 y^2 + z^2 x^2 + z^2 y^2\right),$$

(this is an approximation to the system of 8 identical springs each attached to the vertex of a cube, and all attached to a common particle).
(i) Show V has symmetry $\mathbb{O}_h$ (it is enough to show it is invariant under the 3 given generators).
(ii) Show that the lines $L_1 = \{(0, 0, z) \mid z \in \mathbb{R}\}$ and $L_2 = \{(x, x, 0) \mid x \in \mathbb{R}\}$, and $L_3 = \{(x, x, x) \mid x \in \mathbb{R}\}$, are all 1-dimensional fixed point spaces, and find the corresponding axial subgroups. [Hint: sketch each of these lines on the figure with the cube.]
(iii) Find critical points (equilibria) occurring in these 1-dimensional fixed-point subspaces, and describe how these appear/disappear as λ varies (i.e., the bifurcations involved).
(iv) Find the other 1-dimensional fixed point spaces (all others are equivalent under the summery group $\mathbb{O}_h$ to L_1, L_2 or L_3), and list the corresponding equilibrium points.

