Problems for Chapter 4: Symmetry princple

Qu 4.1: Let $\mathbb{Z}_3 = \{0,1,2\}$ with addition modulo 3, and let $\omega = e^{2\pi i/3}$ (note that $\omega^3 = 1$). Consider the action of $\mathbb{Z}_3$ on the complex plane $\mathbb{C}$ defined by

$$n \cdot z = \omega^n z$$

(i) Show first this is indeed an action. (ii) Show that the equation $z^3 = 8$ has symmetry $\mathbb{Z}_3$ and that the set of solutions also has this symmetry.

Qu 4.2: Let the group G act on two sets X and Y, and suppose that $\phi : X \to Y$ is equivariant. If in addition we suppose ϕ is a bijection, show that $\phi^{-1} : Y \to X$ is also equivariant.

Qu 4.3: Let V be a representation of G, and let $A : V \to V$ be a linear map (a matrix), which is equivariant. Recall that if $\mathbf{v} \neq \mathbf{0}$ satisfies $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$ one says $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.

(i). Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then so is $g \cdot \mathbf{v}$ for each $g \in G$.

(ii). Let E_λ be the λ-eigenspace of A,

$$E_\lambda = \{\mathbf{v} \in V \mid A\mathbf{u} = \lambda\mathbf{u}\}.$$

Show that E_λ is G-invariant.

(iii). Let G_λ be the *generalized eigenspace* of A:

$$G_\lambda = \{\mathbf{v} \in V \mid (A - \lambda I)^n \mathbf{v} = \mathbf{0}\},$$

where $n = \dim V$. Show that G_λ is also G-invariant.

Qu 4.4: Consider the function $f(x, y) = x^2 + y^2 - x^4 - y^4$. Show this is invariant under the group D_4 and find its set $C(f)$ of critical points. Describe how the group acts on this set (i.e., determine the orbits and the orbit type for each orbit), and hence state the Burnside type of the action on $C(f)$.

Qu 4.5: Let V be a representation of G with $V^G = \{0\}$. Prove directly that if $f : V \to \mathbb{R}$ is an invariant function then it has a critical point at 0. [By directly, I mean do not use the Principle of Symmetric Criticality, but you may use its proof to inspire you.]

Qu 4.6: Find all the critical points of the D_3-invariant function $f(x, y) = \lambda(x^2 + y^2) + \frac{1}{3}x^3 - xy^2$. Relate these to the fixed point subspaces for different subgroups of D_3 (refer to Fig. 4.1).

Qu 4.7: For the system of 4 springs discussed in lectures (Example 4.12), study the critical points in the subspace $\mathrm{Fix}(K, \mathbb{R}^2)$, where $K = \langle r_{\pi/4} \rangle$.

Qu 4.8: Let G act on a set X, and let Ω be the set of all functions $f : X \to \mathbb{R}$. Show that the following formula defines an action of G on Ω:

$$(g \cdot f)(x) = f(g^{-1}x), \quad \text{for } f \in \Omega,\ g \in G,\ x \in X.$$

In other words, $g \cdot f = f \circ g^{-1}$.
[Note: The inverse here should be reminiscent of the action by right multiplication of a group on itself, from Chapter 1 (§1.3) which also involves an inverse.]

Qu 4.9: Suppose V, W are representations of a group G. Let $\phi_j : V \to W$ be two equivariant maps, and let $f_j : V \to \mathbb{R}$ be two invariant functions ($j = 1, 2$). Show that the map $\psi : V \to W$ given by

$$\psi(\mathbf{v}) = f_1(\mathbf{v})\,\phi_1(\mathbf{v}) + f_2(\mathbf{v})\,\phi_2(\mathbf{v})$$

is equivariant.

Qu 4.10: Find all homomorphisms of the cyclic group $\mathbb{Z}_4$ to the cyclic group $\mathbb{Z}_6$. [Hint: If H is a cyclic group generated by a, and $\phi : H \to G$ a homomorphism, then ϕ is entirely determined by knowing $\phi(a)$.]

