Problems for Chapter 3: Lattices & Wallpaper Groups

Qu 3.1: Sketch the points (x, y) of the lattice

$$L = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in \mathbb{Z}, x + y \in 2\mathbb{Z}\right\},$$

in the range $-3 \le x \le 3$ and $-3 \le y \le 3$. Show that L is generated by $(1,1)$ and $(2,0)$.

Qu 3.2: Consider the lattice $L = \mathbb{Z}^2$. Show that L can be generated by the vectors $\mathbf{a} = (7,3)$ and $\mathbf{b} = (9,4)$.

Qu 3.3: Extending the previous problem, show that $L = \mathbb{Z}^2$ is generated by integer vectors (a, b) and (c, d) whenever $ad - bc = \pm 1$. [Hint: Consider the matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and show the inverse matrix has integer entries iff $\det A = \pm 1$.]

Qu 3.4: Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Show that they are orthogonal if and only if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

Qu 3.5: Let $L = \left\{(x, y) \in \mathbb{R}^2 \mid y \in \mathbb{Z}, \sqrt{2}(x - y) \in \mathbb{Z}\right\}$. First show L is a subgroup of $\mathbb{R}^2$ (under vector addition). Second show that

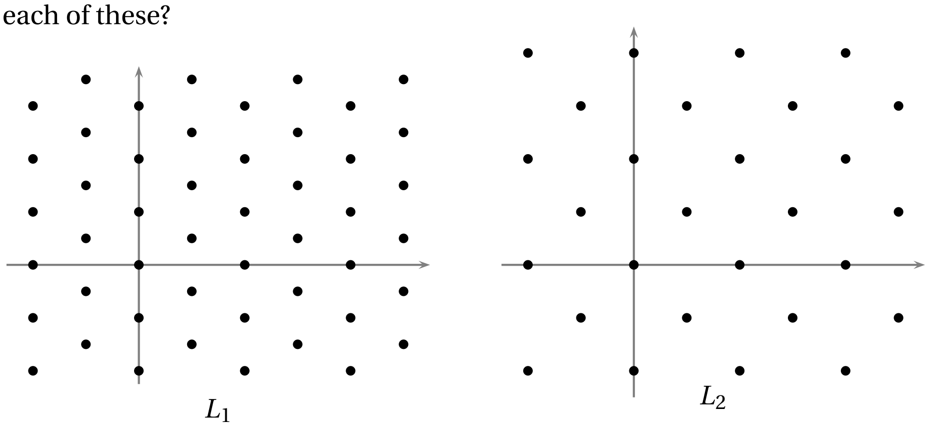
$$L = \left\{ \begin{pmatrix} a + \frac{1}{\sqrt{2}} b \\ a \end{pmatrix} \in \mathbb{R}^2 \mid a, b \in \mathbb{Z} \right\},$$

and hence find two generators of L and deduce that it is a lattice.

Qu 3.6: Consider the two lattices in $\mathbb{R}^2$ defined by,

$$L_1 = \{(2m + n,\ \tfrac{1}{2}n) \mid m, n \in \mathbb{Z}\} \quad \text{and} \quad L_2 = \{(2m + n,\ n) \mid m, n \in \mathbb{Z}\}$$

shown in the figures below. In each case, determine vectors $\mathbf{a}, \mathbf{b}$ according to the conventions, and find the point group. Describe how the point group acts on the lattice. Which of the 5 types of lattice is each of these?



Qu 3.7: Let S be the subset of $\mathbb{R}^2$ consisting of points $(3n+1, 4m-2)$ (with $m, n \in \mathbb{Z}$). Find the set (group) of translations of $\mathbb{R}^2$ preserving the set S; that is, find

$$L = \left\{\mathbf{v} \in \mathbb{R}^2 \mid \mathbf{x}+\mathbf{v} \in S \ \ \forall \mathbf{x} \in S\right\}.$$

[Hint: Let $\mathbf{t_v}(\mathbf{x}) = \mathbf{y}$. Then $\mathbf{y} = \mathbf{x}+\mathbf{v} \in S$, or $\mathbf{v} = \mathbf{y}-\mathbf{x}$ (with $\mathbf{x}, \mathbf{y} \in S$).]

Qu 3.8: [Adapted from past exam] Consider the following subset of the plane:

$$S = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in 2\mathbb{Z},\ x + \tfrac{1}{2}y \in 2\mathbb{Z}\right\}.$$

We wish to show first this is a lattice.

(i). Define the notion of a lattice in the plane.

(ii). Show that the two vectors $\mathbf{u}_1 = (2,0)$ and $\mathbf{u}_2 = (0,4)$ both belong to S, and sketch a diagram showing all the points of S that lie in the rectangle $0 \le x \le 6$ and $0 \le y \le 8$. Deduce that $S \ne \mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(iii). Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $S = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case, and hence deduce that S is a lattice.

(iv). Show that the point group of the lattice S has order 4 by finding appropriate elements of its symmetry group $\mathcal{W}_S < \mathsf{E}(2)$, expressed in the form $(A \mid \mathbf{v})$.

(v). Define a glide reflection. Find a glide reflection $(A \mid \mathbf{v}) \in \mathcal{W}_S$ whose line of reflection is not the line of reflection of a reflection symmetry of S.

Qu 3.9: Let $u > 0$ and consider the 1-dimensional lattice $\mathbb{Z}\{u\}$. Show that the infinite dihedral group $\mathsf{Dih}(\infty)$ (see appendix) acts on this lattice, via

$$a \cdot x = -x, \quad \text{and} \quad b \cdot x = u - x.$$

(You need to show that these two transformations do indeed preserve L and that they satisfy the relations defining $\mathsf{Dih}(\infty)$).)

Qu 3.10: Consider the planar lattice $L = \mathbb{Z}\left\{\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$. Show this is an oblique lattice, and by choosing two appropriate elements, show that it contains a subset which is a rectangular lattice.

Qu 3.11: Which of the 5 types of lattice is $L = \mathbb{Z}\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$?

Qu 3.12: Let $\mathbf{a}, \mathbf{b}$ be two perpendicular vectors of different lengths in $\mathbb{R}^2$, say $|\mathbf{b}| > |\mathbf{a}| > 0$, and let $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ be the resulting rectangular lattice. Let $T \in \mathsf{E}(2)$ be any of the reflections that preserve

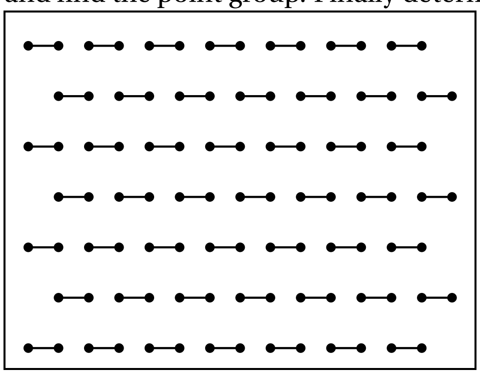
L. Show that there is $\mathbf{v} \in L$ such that either $T = (r_0 \mid \mathbf{v})$ or $T = (r_{\pi/2} \mid \mathbf{v})$. Deduce that the group $\mathcal{W}_L$ of all symmetries of this lattice is generated by $\{\mathbf{t_a}, \mathbf{t_b}, R_\pi, r_0\}$ (why is $r_{\pi/2}$ not needed?).

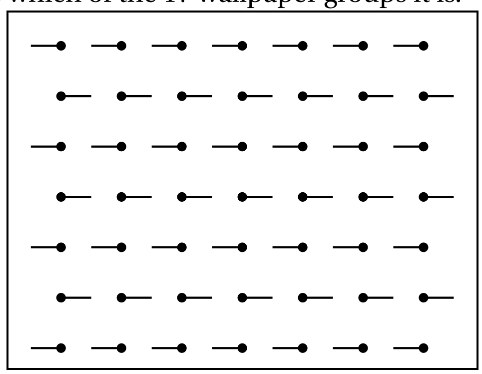
Qu 3.13: Let $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ be any lattice in the plane. There are many possible centres of symmetry: points $\mathbf{c}$ for which a rotation by π about $\mathbf{c}$ (denoted $R_\pi^{\mathbf{c}}$) is a symmetry of the lattice.

(i). Show that $\mathbf{c}_1 = \frac{1}{2}\mathbf{a}$ and $\mathbf{c}_2 = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ are two such points.

(ii). Show that for each centre $\mathbf{c}$ there is a $\mathbf{v} \in L$ such that $R_\pi^{\mathbf{c}} = (R_\pi \mid \mathbf{v}) \in \mathsf{E}(2)$. If $\mathbf{a}$ and $\mathbf{b}$ are such that there are no reflection symmetries and no other rotations, deduce that the group $\mathcal{W}_L$ of all symmetries of this lattice is generated by $\{\mathbf{a}, \mathbf{b}, R_\pi\}$.

Qu 3.14: For each of the following wallpaper patterns, draw generators of the translation lattice and find the point group. Finally determine which of the 17 wallpaper groups it is.





Qu 3.15: Consider the functions of two variables,

$$f(x,y) = \sin(x) + \sin(y) \quad \text{and} \quad g(x,y) \;=\; \sin(x) - 2\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{\sqrt{3}}{2}y\right).$$

The contours of f and g are shown in Figure 1: the lighter, or green, regions are where the function takes positive values and the darker (violet) ones are where the function is negative. Let $\mathcal{W}_f$ and $\mathcal{W}_g$ be their symmetry groups (wallpaper groups).

(a) In each case, find the translation subgroup of $\mathcal{W}$. Which of the 5 types of lattice is this translation subgroup?

(b) Find the point groups J_f and J_g (first just by looking at the diagrams, and then check that these transformations do indeed preserve the function in question).

(c) How is this changed if we allow transformations that change f to $-f$ and g to $-g$? More formally, find the stabilizer of each function under the action of $G = \mathsf{E}(2) \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, -1\}$ and $(T, a) \cdot f = af \circ T^{-1}$, for $T \in \mathsf{E}(2)$ and $a \in \{\pm 1\}$.

Qu 3.16: Refer to Example 3.11, and choose the origin to be at the centre of one of the lozenges. Here we discuss how the group of symmetries is generated. Show that each of the following are in the symmetry group:

$$(R_{\pi/2} \mid \mathbf{e}_1), \quad (r_0 \mid \mathbf{0})$$

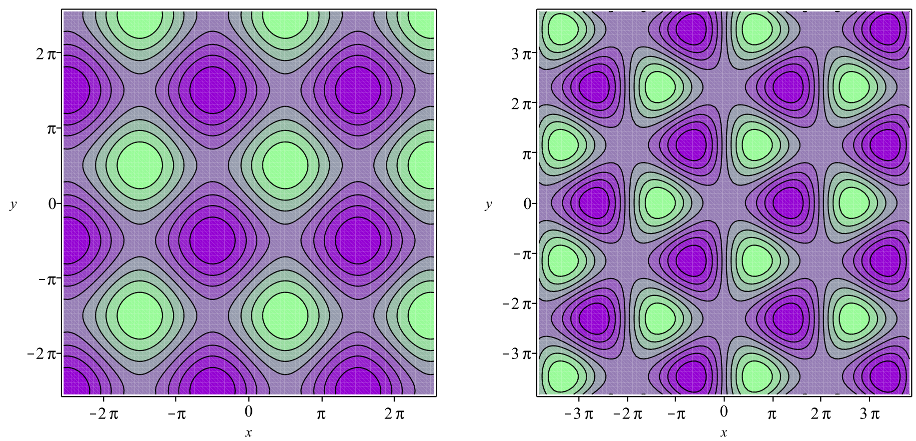


Figure 1: See Problem 3.15. The left-hand figure shows the contours of the function f, the right-hand one the contours of g

where $\mathbf{e}_1 = (1,0)^T$.
(a) Show that the product (composite) $g = (R_{\pi/2} \mid \mathbf{e}_1)(r_0 \mid \mathbf{0})$ is a glide-reflection, and find the line of reflection.
(b) Show that g^2 is one of the vectors that generate the lattice of translations.
(c) Show the other generator is the square of the 'reverse' product $k = (r_0 \mid \mathbf{0})(R_{\pi/2} \mid \mathbf{e}_1)$.
(d) Conclude that the wallpaper group for this pattern is generated by $(R_{\pi/2} \mid \mathbf{e}_1)$ and $(r_0 \mid \mathbf{0})$

Qu 3.17: If we identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$, then there are some famous lattices using the complex numbers:

- Gaussian integers $G = \{a + b\mathrm{i} \mid a, b, \in \mathbb{Z}\} = \mathbb{Z}\{1, \mathrm{i}\}$.
- Eisenstein integers: $E = \{a + b\omega \mid a, b, \in \mathbb{Z}\} = \mathbb{Z}\{1, \omega\}$, where $\omega = \mathrm{e}^{2\pi \mathrm{i}/3} = \frac{1}{2}(-1 + \sqrt{3}\,\mathrm{i})$.

Determine which of the 5 types each of these lattices is. [The interesting thing about these lattices is that they are not only groups, but rings as well, as you can check. For the Eisenstein case, one uses the fact that $\omega^2 + \omega + 1 = 0$.]

Qu 3.18: Here we look at the wallpaper group pgg, or 22×, see Fig. 2. In the figure, there are two types of 'widget': one whose diagonal edge has positive slope and one with negative slope. Call these positive and negative widgets, respectively.

(i). Show on the diagram generators of the lattice of translations, which is a rectangular lattice. Why is it not a centred rectangular lattice?

(ii). Find all centres of rotation (by π).

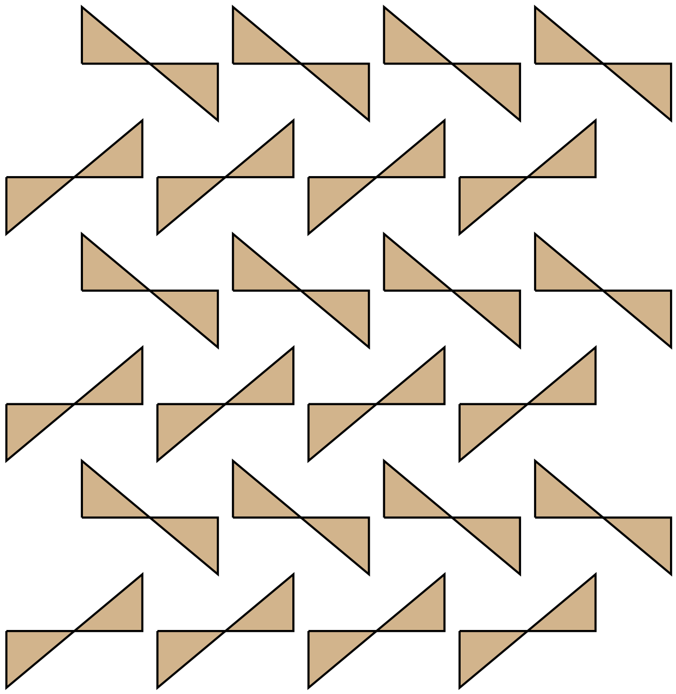


Figure 2: A pattern with symmetry group 22× or pgg, see Problem 3.18

(iii). There are no reflection symmetries of this pattern, but there are both horizontal and vertical glide reflections. Find one of each.

(iv). Let $\mathbf{a}, \mathbf{b}$ be a shortest pair of vectors for the translation lattice, and let $\mathbf{u} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Both by drawing diagrams and by calculation, show that the glide reflections

$$T_1 = (r_0 \mid \mathbf{u}), \quad \text{and} \quad T_2 = (r_{\pi/2} \mid \mathbf{u})$$

belong to the symmetry group of this pattern, where $\mathbf{u} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$.

By considering the squares of these glide-reflections and their products, show that the wallpaper group 22× (or pgg) is generated by T_1 and T_2 (cf. Table 3.2).

Qu 3.19: Find all homomorphisms
(a) from $\mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and
(b) from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. [Hint: If H is a cyclic group generated by a, and $\phi : H \to G$ a homomorphism, then ϕ is entirely determined by knowing $\phi(a)$, because $\phi(a^2) = \phi(a)^2$ etc.]

Qu 3.20† (a) Prove the following lemma:
Let G be a group and $H \lhd G$ a normal subgroup. Then the action of G on itself by conjugation restricts to an action of G on H. Moreover, if H is abelian, this defines an action of the quotient group G/H on H.

(b) Let $G = \mathsf{D}_n$ and $H = \mathsf{C}_n$ (which is a normal subgroup). Determine the resulting action of G/H on H.
(c) Deduce Proposition 3.8 from this lemma.

