Problems for Chapter 2: Euclidean Transformations

Qu 2.1: Deduce from Definition 2.1 that any Euclidean transformation of $\mathbb{R}^n$ is injective.

Qu 2.2: Show the set $\mathsf{O}(n)$ of orthogonal $n \times n$ matrices forms a subgroup of $\mathsf{GL}(n)$. (Recall (see Appendix) that $\mathsf{GL}(n)$ is the group of all invertible $n \times n$ matrices.)

Qu 2.3: Let V be a vector space, which we know is a group under vector addition. For each $\mathbf{v} \in V$ there is the translation $T_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \mathbf{v}$. Show that this defines an action T of V on itself, which coincides with the left translation defined in Section 1.3.

Qu 2.4: Find the eigenvalues of R_θ and r_α. How are the eigenvectors of r_α related to the line of reflection?

Qu 2.5: Verify the identities in Eq. (2.8). Let r_α be a reflection, and find the angle of rotation of $r_\alpha R_\theta r_\alpha^{-1}$. Deduce that, for each n, C_n is a normal subgroup of $\mathsf{O}(2)$.

Qu 2.6: Show that the map $p : \mathsf{E}(n) \to \mathsf{O}(n)$ given by $p(A \mid \mathbf{v}) = A$ is a homomorphism, and deduce that the set of translations in $\mathsf{E}(n)$ is a normal subgroup.

Qu 2.7: In contrast to problem 2.6, show that the map $p' : \mathsf{E}(n) \to \mathbb{R}^n$ defined by $p'(A \mid \mathbf{v}) = \mathbf{v}$ is *not* a homomorphism.

Qu 2.8: Describe the transformation of the plane represented by each of the following Seitz symbols:

$$\text{(i) } (I \mid \mathbf{v}), \quad \text{(ii) } (R_\pi \mid \mathbf{v}), \quad \text{(iii) } (r_{\pi/4} \mid \mathbf{v}), \quad \text{(iv) } (r_0 \mid \mathbf{v}).$$

where $\mathbf{v} = (1,\, 1)^T \in \mathbb{R}^2$.

Qu 2.9: Write the Seitz symbol for each of the following Euclidean transformations of the plane:
(i) the rotation through $\pi/2$ about the point (1,1);
(ii) the reflection in the line $y = x + 1$;
(iii) the glide reflection consisting of the reflection in (ii) followed by a translation by (1,1) (which is parallel to the line of reflection).

Qu 2.10: For $\mathbf{v} \in \mathbb{R}^n$, as usual let $T_{\mathbf{v}}$ denote the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$. Let $A \in \mathsf{O}(n)$. Show that conjugation of a translation by A is another translation, and in particular,

$$A T_{\mathbf{v}} A^{-1} = T_{A\mathbf{v}}.$$

Qu 2.11: Use the Seitz symbol to describe the group $\mathsf{E}(1)$ of Euclidean transformations of the line. [First describe its elements, and then the group structure.]

Qu 2.12: Given any transformation $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, define the 3×3 invertible matrix, written in block form,

$$\psi((A \mid \mathbf{v})) = \left(\begin{array}{c|c} A & \mathbf{v} \\ \hline 0 & 1 \end{array}\right) \in \mathsf{GL}_3(\mathbb{R}).$$

(i) Let $g = (r_{\pi/4} \mid \mathbf{u}) \in \mathsf{E}(2)$, where $\mathbf{u} = (1,1)^T$. Write down $\psi(g)$ and calculate $\psi(g)^2$ and compare with $\psi(g^2)$.
(ii) Show that the map $\psi : \mathsf{E}(2) \to \mathsf{GL}_3(\mathbb{R})$ is a homomorphism.

Qu 2.13: (i) A rotation in the plane through an angle π is called a ***half-turn***. By using the homomorphism $\mathsf{E}(2) \to \mathsf{O}(2)$ taking $(A \mid \mathbf{v})$ to A, show that the composite of two half-turns is a translation.
(ii) For $\mathbf{c} \in \mathbb{R}^2$, denote by $h(\mathbf{c})$ the half-turn with centre $\mathbf{c}$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Express the translation $h(\mathbf{b})h(\mathbf{a})$ explicitly in terms of $\mathbf{a}$ and $\mathbf{b}$.

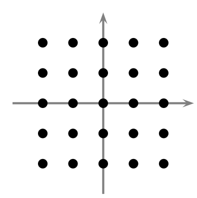
Qu 2.14: Complete the following table, showing the geometric type (point/line ...) of the set of points fixed under each type of Euclidean transformation:

Type	fixed point set
Identity	
Translation	
Rotation	
Reflection	
Glide-reflection	

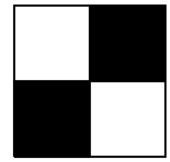
Note that the different geometric types of fixed point set almost distinguishes between the 5 types of transformation.

Qu 2.15: If r_L is the reflection in the line L, and $g \in \mathsf{E}(2)$ is a Euclidean transformation, show that $g\, r_L\, g^{-1}$ is the reflection in the line $g(L)$. [Hint: use Problem 2.14.]

Qu 2.16: Consider the subset $L \subset \mathbb{R}^2$ shown to the right consisting of all points in the plane both of whose coordinates are integers (that is, $L = \mathbb{Z}^2$). Find the rotations and reflections in $\mathsf{O}(2)$ which preserve L (that is, map points in L to points in L).



Qu 2.17: Consider a 2×2 chessboard with 2 black squares and 2 white as shown.



(a) Which reflections and rotations preserve the chessboard with its colouring: i.e. sending black squares to black and white to white. You should use the notation introduced on the sheet on symmetries of the square.
(b) How do we know in advance (i.e., without finding them) that these transformations will form a subgroup of D_4?
[Hint for (b): Consider the action of D_4 on the set $\{C, C'\}$ where C is the chessboard shown, and C' the same board but with the colours swapped.]

Qu 2.18: Continuing the previous question, now consider the action of $\mathbb{Z}_2 = \{e, c\}$ (with $c^2 = e$) on the chessboard where $c \in \mathbb{Z}_2$ acts by changing the colour of every square: black to white and white to black. Combine this with the action of D_4 by rotations and reflections to form an action of the product $G = \mathsf{D}_4 \times \mathbb{Z}_2$. For example, $(R_{\pi/2}, c)$ acts by rotating by $\pi/2$ and then changing the colours. What is the symmetry group of the chessboard, as a subgroup of G? (In other words, which elements of G preserve the chessboard with its colouring?)

Qu 2.19: Find all homomorphisms from $\mathbb{Z}_2$ to each of $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

Qu 2.20† Describe all rotations (centre + angle) and lines of reflection in the plane which preserve the set L defined in Problem 2.16.

Qu 2.21† Find the possible symmetry types of sets of 4 distinct points in the plane.

Qu 2.22† Find the possible symmetry types of quadrilaterals in the plane. (A quadrilateral consists of 4 points *in order*: that is, in $ABCD$ there is no edge joining A to C nor B to D, and as a result the answer should be different to the previous question.)