Solutions for Chapter 1: Group actions

Qu 1.1: Let $\mathbb{Z}_2 = \langle \kappa \rangle$ act on $\mathbb{R}$ by $\rho(\kappa)x = -x$. Find the stabilizers and orbits of each element $x \in \mathbb{R}$.

Solution Let $G = \mathbb{Z}_2$. Then $G_0 = \mathbb{Z}_2$ and for $x \neq 0$, $G_x = \mathbb{1}$. The orbits are: for $x \neq 0$, $G \cdot x = \{x, -x\}$, while $G \cdot 0 = \{0\}$.

Qu 1.2: Suppose two groups G, H both act on a set X via homomorphisms ρ_G and ρ_H respectively, in such a way that, for all $g \in G$, $h \in H$,

$$\rho_H(h) \circ \rho_G(g) = \rho_G(g) \circ \rho_H(h)$$

(one says the two actions *commute*). Show that this gives rise to an action $\overline{\rho}$ of the Cartesian product group $G \times H$ on X, defined by

$$\overline{\rho}(g, h)x = \rho_G(g) \circ \rho_H(h)x.$$

Solution Let $\overline{\rho} : G \times H \to \mathrm{Sym}(X)$ be the map $\overline{\rho}(g, h) = \rho_G(g) \circ \rho_H(h)$. We need to show this is a homomorphism.

Let (g_1, h_1) and (g_2, h_2) be elements of $G \times H$. To show $\overline{\rho}$ is a homomorphism, we need to show that, as permutations,

$$\overline{\rho}((g_1, h_1)(g_2, h_2)) = \overline{\rho}(g_1, h_1) \circ \overline{\rho}(g_2, h_2).$$

This is in fact straightforward: let $x \in X$ then

$$\begin{aligned}
\overline{\rho}((g_1, h_1)(g_2, h_2))(x) &= \overline{\rho}((g_1 g_2,\, h_1 h_2))(x) && \text{(multiplication in Cartesian product)}\\
&= \rho_G(g_1 g_2) \circ \rho_H(h_1 h_2)x && \text{(by definition of the action)}\\
&= \rho_G(g_1)\rho_G(g_2)\rho_H(h_1)\rho_H(h_2)x && \text{(properties of actions)}\\
&= \rho_G(g_1)\rho_H(h_1)\rho_G(g_2)\rho_H(h_2)x && \text{(since actions commute)}\\
&= \overline{\rho}(g_1, h_1) \circ \overline{\rho}(g_2, h_2)(x) && \text{(by definition of the action).}
\end{aligned}$$

Since this is true for all $x \in X$ and all group elements, we have proved that the expressions are indeed equal. A similar calculation shows that $\overline{\rho}\left((g, h)^{-1}\right) = \left(\overline{\rho}(g, h)\right)^{-1}$, and is left to you. It follows that the map $\overline{\rho}$ is a homomorphism.

Qu 1.3: Suppose G is an abelian group and acts on a set X. Show that if $x, y \in X$ lie in the same orbit then their stabilizers are equal: (a) deduce this from Proposition 1.6, and (b) prove it directly.

Solution (a) The proposition says that if[a] $y = g\cdot x$ then $G_y = gG_xg^{-1}$. However, as G is Abelian, $ghg^{-1} = h$ for all $g, h \in G$, so that $gG_xg^{-1} = G_x$. Thus $G_y = G_x$.
(b) From first principles, let $y = g\cdot x$. And suppose $k \in G_x$. Then,

$$k\cdot y = k\cdot(g\cdot x) = kg\cdot x = gk\cdot x = g\cdot(k\cdot x) = g\cdot x = y.$$

Thus $k \in G_x$ implies $k \in G_y$, and we have shown $G_x \subset G_y$. Using the same argument with the roles of x and y reversed shows that $G_y \subset G_x$. Thus indeed, $G_y = G_x$.

[a] when there's no fear of ambiguity, in the solutions we use the more concise notation $g\cdot x$ for the action of g on x, rather than $\rho(g)x$

Qu 1.4: Let G be a finite group acting on the finite set X. For $g \in G$ let X^g denote the subset of X consisting of those elements fixed by g; that is

$$X^g = \{x \in X \mid \rho(g)x = x\}.$$

Suppose g_1 is conjugate to g_2, say $g_2 = hg_1h^{-1}$. Show that $x \in X^{g_1} \iff \rho(h)x \in X^{g_2}$. Deduce that $|X^{g_1}| = |X^{g_2}|$.

Solution Since $g_2 = hg_1h^{-1}$, it follows that $g_1 = h^{-1}g_2h$. Therefore

$$g_1\cdot x = x \iff h^{-1}g_2h\cdot x = x \iff g_2h\cdot x = h\cdot x \iff g_2\cdot(h\cdot x) = h\cdot x. \qquad (*)$$

That is $x \in X^{g_1} \iff h\cdot x \in X^{g_2}$.
There remains to show that the map from X^{g_1} to X^{g_2}, given by $x \mapsto h\cdot x$ is a bijection. This is the case because G is a group and so h has an inverse, namely h^{-1}. Thus,
injective: if $h\cdot x = h\cdot x'$ then multiplying (acting) by h^{-1}, we find $x = x'$,
surjective: if $y \in X^{g_2}$, then let $x = h^{-1}\cdot y$. We need to show $x \in X^{g_1}$. But that follows from the calculation $(*)$ above, since $y \in X^{g_2}$ is equivalent to $x \in X^{g_1}$. This proves the map is a bijection.

Qu 1.5: Draw a picture of a cube, and let $\mathbb{O}$ denote the group of all rotations of the cube (called the 'octahedral group').
(a) How many rotations are there that send the top face to itself? Use the orbit-stabilizer theorem applied to deduce the order of $\mathbb{O}$.
(b) Now use the same theorem applied to a vertex of the cube to check your answer.

Solution (a) Let f be one of the 6 faces of the cube. Now f can be rotated to any of the other faces by an element of the symmetry group $\mathbb{O}$, so the orbit $\mathbb{O}\cdot f$ has 6 elements. On the other hand, there are 4 rotations (including the identity) which takes f to itself; that is the stabilizer $\mathbb{O}_f$ has order 4. Thus, by the orbit-stabilizer theorem, $|\mathbb{O}| = |\mathbb{O}\cdot f|\,|\mathbb{O}_f| = 6\times 4 = 24$.
(b) The argument is similar using vertices: Let A be one of the vertices. There are 8 altogether and $|\mathbb{O}\cdot A| = 8$. There remains to find the stabilizer of the vertex A: this vertex has 3 nearest neighbours, and any rotation that fixes A will cyclically permute these nearest neighbours. This shows that there are 3 rotations fixing A, and hence, by the orbit-stabilizer theorem, $|\mathbb{O}| = |\mathbb{O}\cdot A|\,|\mathbb{O}_A| = 8\times 3 = 24$.
(c) And for a bonus, we could use edges, of which there are 12. [Left to you]

Qu 1.6: For the group D_3 of symmetries of the equilateral triangle (see first example), write down the permutations of D_3 arising as $\lambda(R_{2\pi/3})$ and $\rho(R_{2\pi/3})$, and finally of conjugation by $R_{2\pi/3}$.

Solution $\lambda(R_{2\pi/3})(g) = R_{2\pi/3}g$, and $\rho(R_{2\pi/3})(g) = gR_{2\pi/3}^{-1} = gR_{-2\pi/3}$. Thus, as permutations,

$$\lambda(R_{2\pi/3}) = \begin{pmatrix} I & R_{2\pi/3} & R_{-2\pi/3} & r_0 & r_{\pi/3} & r_{-\pi/3} \\ R_{2\pi/3} & R_{-2\pi/3} & I & r_{\pi/3} & r_{-\pi/3} & r_0 \end{pmatrix},$$

$$\rho(R_{2\pi/3}) = \begin{pmatrix} I & R_{2\pi/3} & R_{-2\pi/3} & r_0 & r_{\pi/3} & r_{-\pi/3} \\ R_{-2\pi/3} & I & R_{2\pi/3} & r_{\pi/3} & r_{-\pi/3} & r_0 \end{pmatrix},$$

and, for the action μ by conjugation,

$$\mu(R_{2\pi/3}) = \begin{pmatrix} I & R_{2\pi/3} & R_{-2\pi/3} & r_0 & r_{\pi/3} & r_{-\pi/3} \\ I & R_{2\pi/3} & R_{-2\pi/3} & r_{-\pi/3} & r_0 & r_{\pi/3} \end{pmatrix}.$$

This shows in particular that the three reflections are all conjugate to one another. [Note: if you write down the permutation $\mu(r_0)$ you will see the two rotations are conjugate.]

Qu 1.7: Consider the action of a group G on itself by conjugation. Use the orbit-stabilizer theorem to prove the *Class Formula*, which states that the number of elements of G that are conjugate to g is equal to $|G|/|C(g)|$, where $C(g)$ is the centralizer of g in G (see the appendix for centralizers).

Solution Fix $g \in G$. Now, for $k \in G$,

$$\mu(k)g = kgk^{-1}.$$

First find the stabilizer of g:

$$G_g = \{k \in G \mid kgk^{-1} = g\}.$$

Note that $kgk^{-1} = g$ is equivalent to $kg = gk$, that is $k \in C(g)$. Thus the stabilizer of g is the set of elements that commute with g:

$$G_g = C(g).$$

On the other hand, the orbit $G \cdot g$ is the set of all elements conjugate to g. The orbit-stabilizer theorem therefore implies the number of points in the orbit (or number of elements conjugate to g) is equal to $|G|/|C(g)|$.

Qu 1.8: Let $\rho : G \to S(G)$ be the action by right multiplication defined in lectures: $\rho(g)(h) = hg^{-1}$. Show this is an action, but that in general the map $\rho' : G \to S(G)$ given by $\rho'(g)(h) = hg$ is not an action.

Solution The action by right multiplication is $\rho(g)(k) = kg^{-1}$. We need to check that $\rho(gh) = \rho(g)\rho(h)$ (see the Appendix if in doubt). Now, for all $k, g, h \in G$, we have

$$\rho(gh)(k) = k(gh)^{-1} = kh^{-1}g^{-1}.$$

On the other hand,

$$\rho(g) \circ \rho(h)(k) = \rho(g)(kh^{-1}) = kh^{-1}g^{-1}.$$

These are equal so it is indeed an action.
Question: Where does this argument fail if we use g on the right rather than g^{-1}?

Qu 1.9: Suppose G acts on a set X. (a) Show that the action is effective if and only if the homomorphism $\rho : G \to S(X)$ is injective. (b) Show that an action is transitive if and only if there is only one orbit, X itself

Solution (a) The action is effective means that $\forall g \neq e$ there is an $x \in X$ such that $g \cdot x \neq x$, or in other words $\forall g \neq e$ the permutation $\rho(g) \neq e$ (is not the identity permutation). Recall that $\ker \rho = \{g \in G \mid \rho(g) = e\}$. Thus, effective means that $\ker \rho = \{e\}$, which is to say ρ is injective.
(b) If there are more than one orbit, then let x be in one orbit and y in a different orbit. It follows that there is no $g \in G$ for which $g \cdot x = y$, so the action is not transitive. Conversely, if there is only one orbit then for any $x, y \in X$ there is a $g \in G$ for which $y = g \cdot x$, showing the action is transitive.

Qu 1.10: Suppose G acts on X and let $x \in X$ and $H = G_x$. Suppose H' is conjugate to H. Show that there is a point $y \in X$ with stabilizer H'. Deduce from the action of D_4 on the vertices of a square that r_0 and $r_{\pi/4}$ are not conjugate in D_4.

Solution We want to find an element $y \in Y$ for which $G_y = H'$. As H' is conjugate to H, there is a $g \in G$ such that $H' = gHg^{-1}$. Now refer to Proposition 1.6, and let $y = g \cdot x$. The proposition tells us that $G_y = gG_xg^{-1}$, and since $G_x = H$ it follows that $G_y = H'$.
For the action of D_4 on the vertices of the square $V = \{A, B, C, D\}$, note that $r_{\pi/4}$ fixes A, while r_0 does not fix any points; that is, $G_A = \langle r_{\pi/4} \rangle$, whereas there is no point whose stabilizer is $\langle r_0 \rangle$.

Qu 1.11: Suppose a group G acts on a finite set X, and let $\mathcal{P}(X)$ denote the power set of X (the collection of all subsets of X— there are $2^{|X|}$ of them in all).

(i). Show that the following formula defines an action of G on $\mathcal{P}(X)$:

$$g \cdot S = \{g \cdot x \mid x \in S\} \quad (g \in G,\ S \subset X).$$

(ii). Let $\kappa : \mathcal{P}(X) \to \mathcal{P}(X)$ be the 'complement map', $\kappa(S) = S' = X \setminus S$. Show that κ is a bijection and is equivariant.

(iii). For $k = 0, \dots, |X|$ denote by $\mathcal{P}(X)_k$ the collection of those subsets of X with cardinality k. Use the map κ from above to show that the action of G on $\mathcal{P}(X)_k$ is isomorphic to the action on $\mathcal{P}(X)_{n-k}$, where $n = |X|$.

(iv). Show that together, the actions of G and $\mathbb{Z}_2 = \langle \kappa \rangle$ define an action of $G \times \mathbb{Z}_2$ on $\mathcal{P}(X)$ (see Problem 1.2 above).

Solution (i). If $S \subset X$ (ie, $S \in \mathcal{P}(X)$), clearly $g \cdot S$ is also a subset of X (ie, $g \cdot S \in \mathcal{P}(X)$), so we just need to show the homomorphism property: $g \cdot (h \cdot S) = (gh) \cdot S$. Now,

$$g \cdot (h \cdot S) = g \cdot \{h \cdot x \mid x \in S\} = \{g \cdot (h \cdot x) \mid x \in S\}.$$

Moreover, since we have an action of G on X, it follows that $g \cdot (h \cdot x) = (gh) \cdot x$. Therefore

$$g \cdot (h \cdot S) = \{(gh) \cdot x \mid x \in S\} = gh \cdot S$$

as required.
(ii). κ is a bijection: (a) it is injective because if S and T are two subsets of X and $S' = T'$ then $S = T$. (b) it is surjective because given any subset $T \subset X$ there is an S such that $T = S'$, namely $S = T'$ (the complement of the complement is the set itself, or $\kappa^2 = e$). To show it is equivariant, let $g \in G$. Then

$$\kappa(g \cdot S) = \{g \cdot x \mid x \in S\}' = \{g \cdot x \mid x \notin S\}$$

(this last equality follows because g is a bijection/permutation), and

$$g \cdot \kappa(S) = g \cdot S' = \{g \cdot x \mid x \notin S\}.$$

Therefore, for all g and S, $\kappa(g \cdot S) = g \cdot (\kappa(S))$, which is to say that κ is equivariant.
(iii) First note that G does indeed act on each $\mathcal{P}(X)_k$, because $g \cdot S$ and S have the same cardinality. Now, if $S \in \mathcal{P}(X)_k$ then $\kappa(S) \in \mathcal{P}(X)_{n-k}$. Moreover

$$\kappa : \mathcal{P}(X)_k \longrightarrow \mathcal{P}(X)_{n-k}$$

is a bijection and is equivariant (as proved in (2)), and therefore defines an isomorphism of actions of G.
(iv) We have already pointed out that $\kappa^2 = e$, the identity (that is, $\kappa(\kappa(S)) = (S')' = S$). It therefore defines an action of $\mathbb{Z}_2$. We have already shown that κ commutes with the action of G (that is, $\kappa \circ g = g \circ \kappa$), and hence by Problem 1.2 above, we obtain an action of $G \times \mathbb{Z}_2$, defined by,

$$\begin{cases} (g,e) \cdot S &= g \cdot S \\ (g,\kappa) \cdot S &= g \cdot S'. \end{cases}$$

Qu 1.12: Let H be a subgroup of K and K a subgroup of G; that is $H < K < G$. Show that the map

$$\pi : G/H \to G/K, \quad \pi(gH) = gK$$

is equivariant. (Here the actions of G on G/H and G/K are λ_H and λ_K, as defined in Section 1.4.)

Solution This is just an issue of understanding the question — the proof is trivial! The action of G on G/H is given by $\lambda_H(h)(gH) = (hg)H$. Now

$$\pi(\lambda_H(gH)) = \pi(hgH) = hgK = h(gK) = \lambda_K(h)\pi(gK).$$

Thus we have shown $\pi \circ \lambda_H(h) = \lambda_K(h) \circ \pi$, as required for equivariance.

Qu 1.13: Let H and K be two subgroups of a group G, and suppose $\psi : G/H \to G/K$ is a G-equivariant map for the actions λ_H and λ_K respectively.
(a) Show that ψ is surjective.
(b) Let $g_0 \in G$ be such that $\psi(H) = g_0K$. Show that $H < g_0Kg_0^{-1}$.

Solution (a) Any element of G/K is a coset of the form gK (for some $g \in G$). Now let g_0 be such that $\psi(H) = g_0K$, and let $g_1 = gg_0^{-1}$. Then

$$\psi(g_1H) = g_1\psi(H) = g_1g_0K = gK$$

as required (the fist equality here uses the given fact that ψ is equivariant). Thus gK is in the image of ψ. But gK is an arbitrary coset of K, so that ψ is indeed surjective.
(b) This is trickier! Now $hH = H$ for any $h \in H$. But by equivariance, $\psi(H) = \psi(hH) = h\psi(H) = hg_0K$. Therefore, $g_0K = hg_0K$. Equivalently, $g_0^{-1}hg_0K = K$. This implies that $g_0^{-1}hg_0 \in K$, or equivalently, $h \in g_0Kg_0^{-1}$.
But this is true for each $h \in H$, and consequently $H \subset g_0Kg_0^{-1}$.

Qu 1.14: Suppose a group G acts on two sets X and Y. Consider the set $\mathrm{Map}(X,Y)$ of all maps from X to Y. Define an action ρ_M of G on this set of maps, by putting $\rho_M(g)\phi = \psi$, where

$$\psi = \rho_Y(g)\circ\phi\circ\rho_X(g)^{-1},$$

or more explicitly, $(\rho_M(g)\phi)(x) = \rho_Y(g)\phi(\rho_X(g^{-1})x)$ for $\phi\in\mathrm{Map}(X,Y)$ (and note that $\rho_X(g^{-1}) = \rho_X(g)^{-1}$). Show first that ρ_M is indeed an action on $\mathrm{Map}(X,Y)$, and second that $\phi\in\mathrm{Map}(X,Y)$ is fixed by all of G if and only if ϕ is equivariant.

Solution We need to show the homomorphism property. Let $\rho_M : G\to S(\mathrm{Map}(X,Y))$ be as given. We need to show $\rho_M(gh) = \rho_M(g)\circ\rho_M(h)$ (we will emphasize the binary law as composition in the answer).
Then $\rho_M(gh)(\phi) = \rho_Y(gh)\circ\phi\circ\rho_X(gh)^{-1}$. Moreover, $\rho_M(h)(\phi) = \rho_Y(h)\circ\phi\circ\rho_X(h)^{-1}$, and therefore

$$\rho_M(g)\circ\rho_M(h)(\phi) = \rho_Y(g)\circ\rho_Y(h)\circ\phi\circ\rho_X(h)^{-1})\circ\rho_X(g)^{-1}.$$

Thus expanding $\rho_M(gh) = \rho_M(g)\circ\rho_M(h)$ (we have actions on X) the equality follows.
The final part is just a reading of the definition of equivariance.

Qu 1.15: (a) Show the map ϕ given in Example 1.13 is equivariant, and hence deduce as claimed that the actions of D_3 on V and E are isomorphic.
(b) Now consider a square with vertices $V = \{A,B,C,D\}$ and edges $E = \{e,f,g,h\}$, and let D_4 act on this square. By considering the fixed points of the elements of the group show that the actions on V and E are *not* isomorphic.

Solution (a) We only need check equivariance on generators of the group. Now D_3 is generated by r_0 and $R_{2\pi/3}$, so we use those. Now r_0 gives the permutations

$$\begin{pmatrix} A & B & C\\ A & C & B\end{pmatrix},\quad\text{and}\quad\begin{pmatrix} BC & AC & AB\\ BC & AB & AC\end{pmatrix}.$$

Thus $r_0(\phi(A)) = r_0(BC) = BC$ while $\phi(r_0(A)) = \phi(A) = BC$. Similarly, $r_0(\phi(B)) = r_0(AC) = AB$ while $\phi(r_0(B)) = \phi(C) = AB$. A similar calculation verifies the remaining equations.
(b) The element $r_{\pi/4}$ has two fixed points in V, but none in E. That is enough to show the actions are not isomorphic.

Qu 1.16: Consider the action of the group D_4 on the set of 21 points shown in Fig. 1 below. (i) Determine its Burnside type. (ii) Verify the orbit counting theorem for this action of D_4.

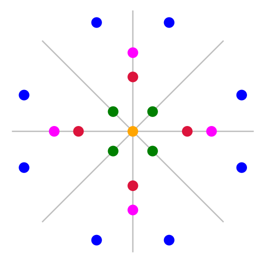

Figure 1

Solution The 21 points divide into 5 orbits (see the different colours). They are as follows.
• The 8 blue points form a single orbit, with trivial stabilizer $\mathbb{1}$.
• The 8 pink/red points (2 different shades) form 2 orbits, and each point has stabilizer either $\langle r_0\rangle$ or $\langle r_{\pi/2}\rangle$. These are conjugate subgroups of D_4, and thus the orbit type is (D_1), where as usual $\mathsf{D}_1 = \langle r_0\rangle$.
• The 4 green points form a single orbit, but each point has a point with reflection symmetry, in particular the orbit type is (D_1'), where $\mathsf{D}_1' = \langle r_{\pi/4}\rangle$.
• The yellow point at the origin is the final orbit, with orbit type (D_4).
(i) Thus the Burnside type of the action is $(\mathsf{D}_4) + (\mathsf{D}_1') + 2(\mathsf{D}_1) + (\mathbb{1})$.
(ii) For each element $g \in \mathsf{D}_4$ we consider the number of points fixed:

g	e	R	R^2	R^3	r_0	$r_{\pi/2}$	$r_{-\pi/4}$	$r_{\pi/4}$
$\lvert X^g\rvert$	21	1	1	1	5	5	3	3

Thus $\sum_{g\in\mathsf{D}_4} |X^g| = 40$. The group is of order 8, and hence

$$|X/G| \;=\; \tfrac{1}{8}40 \;=\; 5,$$

which is indeed equal to the number of orbits.

Qu 1.17: Consider the right action of a subgroup K on a group G. Show that the orbits of this action are the left cosets of K, and deduce that the set of orbits is G/K (thereby showing the notation of cosets is compatible with the notation of orbit space).

Solution The right action of K on G is $\rho(k)(g) = gk^{-1}$. The orbit of g under this action is the set

$$\{\rho(k)(g) \mid k \in k\}.$$

But this is equal to

$$\{gk^{-1} \mid k \in k\} = \{gk^{-1} \mid k^{-1} \in K\} = gK,$$

(using the fact that if $k \in K$ then $k^{-1} \in K$).

Qu 1.18: Consider a disk divided into 6 equal sectors. You have 3 colours at your disposal to colour the 6 segments. Find the number of distinct colourings there are, where (a) distinct means 'up to rotation', and (b) it means 'up to rotation and reflection'.

Solution Since there are 6 sectors and 3 colours, there are $3^6 = 729$ possible colourings altogether. Let X be the set of all possible coloured disks. Thus $|X| = 729$.
(a) The group $G = \mathsf{C}_6 = \langle R \rangle$ of rotations acts on this disk and hence on the set X (here $R = R_{2\pi/6}$). We need to know how many orbits there are. To use the orbit counting theorem, we need to go through each of the elements in turn and see how many colourings are fixed by that element. For example, if $g = I$ then $X^g = X$. If $g = R$ then X^R consists of those colourings which don't change under a rotation of order 6, but this means all the sectors are coloured the same. Thus we find $|X^R| = 3$ (for the 3 possible colours). For X^{R^2}, sectors 1, 3 and 5 must be coloured the same, and sectors 2, 4 and 6 must be coloured the same. Thus we only ned to choose the colours of sectors 1 and 2, giving $|X^{R^2}| = 3^2 = 9$. Continuing in this way gives,

$$|X/\mathsf{C}_6| = \frac{1}{6}(729 + 3 + 9 + 27 + 9 + 3) = 130.$$

(b) Now the group is extended to include reflections, so $G = \mathsf{D}_6$. The calculations for the rotations are the same as above, but the reflections fall into two types. Of the 6 reflections, 3 allow 3 different colours, while the other 3 allow 4 distinct colours (why the difference?). Then

$$|X/\mathsf{D}_6| = \frac{1}{12}\big(729 + 3 + 9 + 27 + 9 + 3 + 3 \times (3^3 + 3^4)\big) = 92.$$

Qu 1.19† Suppose the group G acts on the set X and let H be a subgroup of G. Denote by X_H the subset $X_H = \{x \in X \mid G_x = H\}$. Let $x \in X_H$. Show that $g \cdot x \in X_H$ if and only if $g \in N_G(H)$ (the normalizer of H in G).

Solution Let $y = g \cdot x$. Then $G_y = gG_xg^{-1}$ (see Proposition 1.6). We require $y \in X_H$, which means $G_y = G_x$. That is, $H = gHg^{-1}$, which in turn means $g \in N_G(H)$.

Qu 1.20† Consider first the set $G \times X$, with an action of H given by

$$\sigma(h)(g,x) \;=\; (gh^{-1}, h\cdot x),$$

where $h\cdot x$ denotes the given action of H on X (alternatively, you can write $\rho_X(h)x$). Now define $Y = G \times_H X$ to be the quotient of $G \times X$ by this H-action. Thus, an element $[g,x] \in Y$ is an equivalence class,

$$[g,x] \;=\; \left\{(gh^{-1}, h\cdot x) \mid h \in H\right\} \;\subset\; G \times X.$$

(i). Show $[e, h_0 \cdot x] = [h_0, x]$ (for all $h_0 \in H$ and $x \in X$).

(ii). Show that the formula

$$\rho_Y(g)[g', x] := [gg', x]$$

defines a well-defined action ρ_Y of G on Y.

(iii). Show that the map $\phi : X/H \to Y/G$ defined by $\phi(H\cdot x) = G\cdot[e,x]$ is well-defined, and defines a bijection between X/H and Y/G.

Solution (i) Let $h_0 \in H$, $x \in X$. Then $[h_0, x] = [h_0 h^{-1}, h\cdot x]$ (for all $h \in H$, by definition of the equivalence classes). In particular, if we put $h = h_0$ we find $[h_0, x] = [h_0 h_0^{-1}, h_0 \cdot x] = [e, h_0 \cdot x]$ as required.

(ii) We first need to show ρ_Y is well-defined. To this end, consider two elements of the equivalence class $[g_1, x_1]$, namely $p_1 = [g_1, x_1]$ and $p_2 = [g_2, x_2]$, where $g_2 = g_1 h_1^{-1}$ and $x_2 = h_1 x_1$ (for some $h_1 \in H$). Then

$$\rho_Y(g)p_1 = [gg_1, x_1] = \{(gg_1h^{-1}, \rho_X(h)x_1) \mid h \in H\}$$

Similarly,

$$\begin{aligned}
\rho_Y(g)p_2 &= [gg_2, x_2] \\
&= \{(gg_2h^{-1}, \rho_X(h)x_2) \mid h \in H\} \\
&= \{(g(g_1h_1^{-1})h^{-1}, \rho_X(h)\rho_X((h_1)x_1) \mid h \in H\} \\
&= \{(gg_1(hh_1)^{-1}), \rho_X(hh_1)x_1) \mid h \in H\} \\
&= \{(gg_1k^{-1}, \rho_X(k)x_1) \mid k \in H\} \quad \text{where } k = hh_1 \in H \\
&= \rho_Y(g)p_1,
\end{aligned}$$

as required. It is easy to show it's an action (left to you).

(iii) In principle, the definition depends on x rather then $H\cdot x$. So suppose $H\cdot x = H\cdot y$. Then $y = h_0 \cdot x$ for some $h_0 \in H$.

$$\begin{aligned}
\phi(H\cdot y) &= G\cdot[e, y] \\
&= G\cdot[h_0, h_0^{-1}\cdot y] \;=\; G\cdot[h_0, x] \\
&= G\cdot[e, x] \;=\; \phi(H\cdot x),
\end{aligned}$$

because $h_0^{-1}[h_0, x] \in G\cdot[h_0, x]$ and $h_0^{-1}[h_0, x] = [h_0^{-1}h_0, x] = [e, x]$. Thus ϕ is well-defined.

Surjective: This is straightforward: let $[g, y] \in Y$. We want to show $G\cdot[g, y]$ is in the image of ϕ. But $\phi(H\cdot y) = G\cdot[e, y]$ (by definition of ϕ) and this contains $[g, y]$, so we're done.

Injective: Suppose $\phi(H\cdot x) = \phi(H\cdot y)$. Then $[e, x] \in G\cdot[e, y]$, or $[e, x] = [g, y]$ for some $g \in G$. In particular, this means $(e, x) = (gh^{-1}, h\cdot y)$ for some $h \in H$ (by the definition of $[g, y]$). And therfore $x = h\cdot y$, which implies that $H\cdot x = H\cdot y$, as required for injectivity.

Qu 1.21† Let H and K be two subgroups of a group G. A subset of G of the form

$$HgK := \{hgk \mid h \in H,\ k \in K\} \subseteq G$$

is called a ***double coset***. The set of all such double cosets is denoted $H\backslash G/K$.

(i). The action λ_K of G on G/K (defined in §1.4) restricts on an action of H. Show that each double coset can be identified with an orbit of this action of H on G/K.

(ii). Let $G = S_3$, with H, K being the subgroups of order 2 given by

$$H = \langle (1\,2) \rangle, \quad K = \langle (1\,3) \rangle.$$

List the double cosets of S_3. [Hint: there are just 2 and they are not of the same size.]

(iii). Now let $G = S_4$, and let H, K be as above. How many double cosets are there?

Solution (i) The left cosets of K are subsets of the form $gK \in G/K$. The action by H is

$$\lambda_K(h)(gK) = hgK.$$

This is for $h \in H$. An orbit of this action is a set of the form $\{hgK \mid h \in H\}$, which is precisely the double coset HgK. That is, the double coset D_g is the H-orbit of the left coset gK (or more precisely, it is the union of the left cosets in the H-orbit of gH). (ii) There are 3 left cosets of K, namely

$$\{e, (1\,3)\}, \quad \{(1\,2), (1\,3\,2)\}, \quad \text{and} \quad \{(2\,3), (1\,2\,3)\}.$$

The action of H on this set of 3 cosets is just determined by multiplying on the left by $(1\,2) \in H$. This exchanges the first two cosets and leaves the third alone (eg, $(1\,2)e = (1\,2)$ and $(1\,2)(1\,3) = (1\,3\,2)$: remember we compute products of cycles from right to left). Thus there are two orbits:

$$\Big\{\{e, (1\,3)\}, \{(1\,2), (1\,3\,2)\}\Big\}, \quad \text{and} \quad \Big\{\{(2\,3), (1\,2\,3)\}\Big\}.$$

The double cosets are obtained by just taking the sets of elements occurring in each orbit; there are therefore two double cosets, one for each orbit:

$$\Big\{e, (1\,3), (1\,2), (1\,3\,2)\Big\}, \quad \text{and} \quad \Big\{(2\,3), (1\,2\,3)\Big\}.$$

(iii) In this example, $|S_4| = 4! = 24$, so in G/K there are $24/2 = 12$ left cosets. One finds — by the same procedure as above — that there are 7 double cosets. They are,

$$\Big\{e, (1\,3), (1\,2)\,(1\,3\,2)\Big\}, \quad \Big\{(2\,3), (1\,2\,3)\Big\}, \quad \Big\{(2\,4\,3), (1\,2\,4\,3)\Big\},$$

$$\Big\{(1\,4), (1\,3\,4), (1\,4\,2), (1\,3\,4\,2)\Big\}, \quad \Big\{(2\,4), (1\,3)(2\,4), (1\,2\,4), (1\,3\,2\,4)\Big\},$$

$$\Big\{(3\,4), (1\,4\,3), (1\,2)(3\,4), (1\,4\,3\,2)\Big\}, \quad \text{and} \quad \Big\{(1\,4)(2\,3), (1\,2\,3\,4), (2\,3\,4), (1\,4\,2\,3)\Big\}.$$

Notice that of the 7 double cosets there are 2 with two elements and 5 with four elements: unlike ordinary (single) cosets, these are not all of the same size.
(†) *Extra problem: relate the size of the double coset HgK to the size of the intersection $H \cap gKg^{-1}$.*