2 hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN NATURE

01 June 2017

09:45 – 11:45

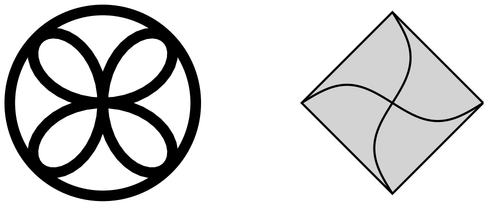
Answer **ALL FOUR** questions

Electronic calculators may be used, provided that they cannot store text, transmit or receive information, or display graphics

---

1.

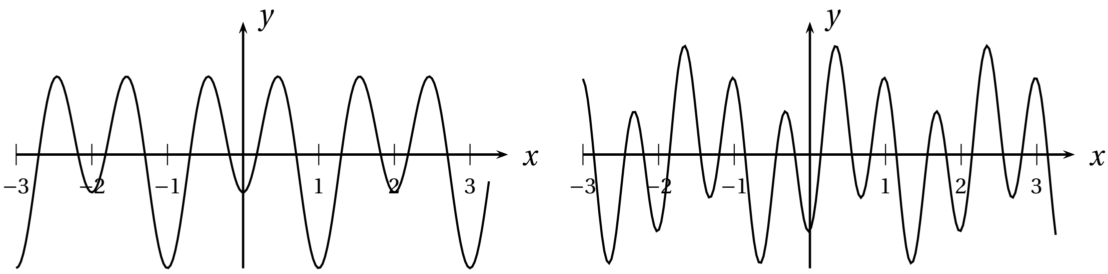
(a) Name the symmetry groups of the following two planar shapes, and state in each case the order of the group, generator(s) of the group and whether the group is Abelian.



(i) (ii)

[6 marks]

(b) Consider the two functions $f(x) = \cos(\pi x) - 2\cos(2\pi x)$ and $g(x) = \sin(\pi x) - 2\cos(3\pi x)$. Their graphs are shown below.



$y = f(x)$ $y = g(x)$

Determine geometrically the full symmetry group of each of the functions f and g, including possible transformations of x that reverse the sign of the function, and verify these are symmetries using the expressions for the functions.

[8 marks]

2.

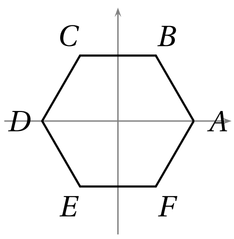
(a) Suppose a finite group G acts on a set X.

(i) Define the orbit and stabilizer of a point $x \in X$. Show that the stabilizer of a point is always a subgroup of G.

(ii) Let $y = k \cdot x$ for some $k \in G$. Show that the stabilizer subgroups G_x and G_y are conjugate subgroups of G.

[6 marks]

(b) Let D_6 be the usual dihedral group of order 12, and let $V = \{A, B, \ldots, F\}$ be the set of vertices of the hexagon as shown on the right. Denote by $\mathscr{P}(V)$ the power set of V, which is the set of all 2^6 subsets of V. Denote by $\mathscr{P}(V)_k$ the subset of $\mathscr{P}(V)$ consisting of those subsets of cardinality k ($k = 0, 1, \ldots, 6$).



There is a natural action of D_6 on $\mathscr{P}(V)$ defined by

$$g \cdot S = \{g \cdot x \mid x \in S\}, \qquad \text{for } S \subset V,\ g \in \mathsf{D}_6.$$

(i) State the orbit stabilizer theorem, and for $S_1 = \{B, C\}$ find the stabilizer and orbit of S_1 in $\mathscr{P}(V)_2$, and show they satisfy the theorem.

(ii) Find an element $S_2 \in \mathscr{P}(V)_2$ whose stabilizer has different order to that of S_1.

Consider in addition the action of $\mathbb{Z}_2 = \{e, \sigma\}$ on $\mathscr{P}(V)$ defined by

$$\sigma(S) = S'$$

where S' is the complement of S in V. Let $G = \mathsf{D}_6 \times \mathbb{Z}_2$.

(iii) Write down $\sigma(S_1)$. Use σ to show that the D_6 actions on $\mathscr{P}(V)_2$ and $\mathscr{P}(V)_4$ are isomorphic.

(iv) Consider $S_0 = \{A, C, E\}$ and find its stabilizer for each of the D_6 and the $\mathsf{D}_6 \times \mathbb{Z}_2$ actions on $\mathscr{P}(V)$. Show that the stabilizer for the $\mathsf{D}_6 \times \mathbb{Z}_2$ action is the graph of a homomorphism from D_6 to $\mathbb{Z}_2$. What is the relevance of the kernel of this homomorphism?

[16 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, where $A \in \mathsf{O}(2)$ and $\mathbf{v} \in \mathbb{R}^2$, and deduce that multiplication in the group is given by

$$(A \mid \mathbf{v})(B \mid \mathbf{u}) = (AB \mid \mathbf{v} + A\mathbf{u}).$$

[4 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in \mathbb{Z},\ x + y \in 2\mathbb{Z}\right\}.$$

(i) Show that the two vectors $\mathbf{u}_1 = (2, 0)$ and $\mathbf{u}_2 = (0, 2)$ both belong to L, and sketch a diagram showing all the points of L that lie in the rectangle $0 \le x \le 6$ and $0 \le y \le 6$. Deduce that $L \ne \mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case.

(iii) Define the point group of a lattice. Show that the point group J_L of the lattice L has order 8 by finding appropriate elements of its symmetry group $\mathscr{W}_L < \mathsf{E}(2)$, expressed in the form $(A \mid \mathbf{v})$.

(iv) Now consider the wallpaper pattern produced by placing the symbol $\bigcup$ at those points of L where $y \in 2\mathbb{Z}$, and $\bigcap$ at the points of L where $y \notin 2\mathbb{Z}$. Let $\mathscr{W}$ be the wallpaper group of this pattern, with point group J. Draw the modified version of the diagram from (i) (with just $0 \le x \le 4$ and $0 \le y \le 4$), and show that $J = \mathsf{D}_2$. Explain how r_0 is an element of the point group, again by finding an element $(r_0 \mid \mathbf{v}) \in \mathscr{W}$. Describe this last transformation.

[20 marks]

4.

(a) Let $\mathbb{Z}_2 = \{e, \beta\}$ act on $\mathbb{R}$ by $\beta \cdot x = -x$. Show that the differential equation $\dot{x} = x\cos(x)$ has symmetry group $\mathbb{Z}_2$. Let $x(t)$ be the solution with initial value $x(0) = 1$, and let $u(t)$ be the solution with initial value $u(0) = -1$. How are $x(t)$ and $u(t)$ related? [4 marks]

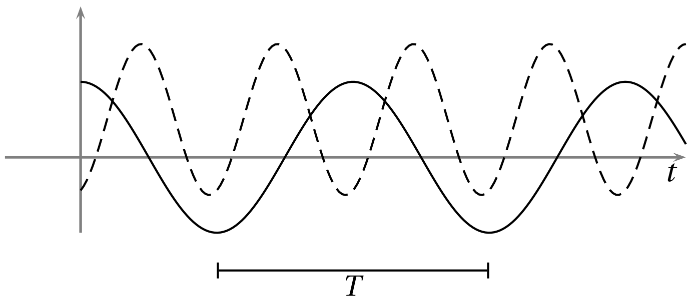
(b) Let D_4 be the usual dihedral subgroup of order 8 of $\mathsf{O}(2)$, generated by r_0 and $R_{\pi/2}$. List all axial subgroups of D_4. Show that the smooth function $f(x, y) = x^2 + y^2 - x^2 y^2$ is invariant under the D_4 action. Now consider the system of differential equations

$$\dot{x} = \frac{\partial f}{\partial x}, \qquad \dot{y} = \frac{\partial f}{\partial y}.$$

Find all equilibria of this system with axial symmetry, and find the solution with initial condition $(x, y) = (1, 0)$. [8 marks]

(c) Suppose $\gamma(t)$ is a periodic solution with fundamental period T of a differential equation with symmetry group G. Define the symmetry group of the periodic solution.

Suppose a mechanical system in the plane has potential energy $V(x, y)$, and that the system has D_4 symmetry. Let γ be a periodic orbit with period T and symmetry group $\widetilde{\mathsf{C}_4}$, and let $\delta(t)$ be a periodic solution of the same period and with symmetry $\widetilde{D_1}$. Write down in each case what the symmetry group tells us about the periodic orbit. Below is shown numerical data taken from an experiment with this symmetry showing both $x(t)$ and $y(t)$. Determine whether the solution has symmetry $\widetilde{D_1}$ or $\widetilde{C_4}$, explaining which graph is $x(t)$ and which is $y(t)$.



[8 marks]

END OF EXAMINATION PAPER