2 hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN NATURE

20 May 2016

14:00 – 16:00

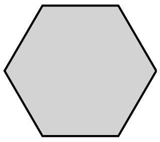
Answer **ALL FOUR** questions

Electronic calculators may be used, provided that they cannot store text, transmit or receive information, or display graphics

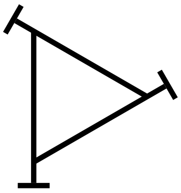
---

1.

(a) Name the symmetry groups of the following two planar shapes, and state in each case the order of the group and whether the group is Abelian.



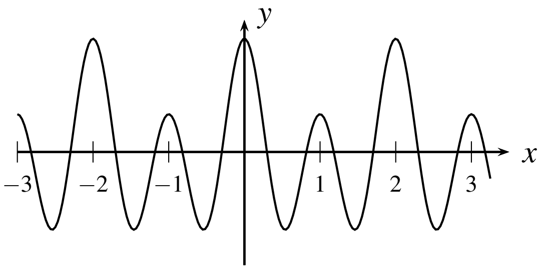
(i)



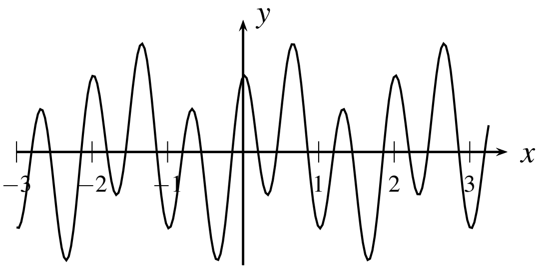
(ii)

[6 marks]

(b) Consider the two functions f and g of one variable whose graphs are shown below.



$y = f(x)$



$y = g(x)$

Describe the full symmetry group of each of the functions f and g, including possible transformations that reverse the sign of the function. (It is sufficient to give generators of the groups, making clear which if any reverse the sign of the function.)

[8 marks]

2.

(a) Suppose a finite group G acts on a set X.

(i) Define the orbit and stabilizer of a point $x \in X$. Show that the stabilizer of a point is always a subgroup of G.

(ii) Let $h \in G$ and $y = h \cdot x$. Show that the subset

$$G_{x,y} = \{g \in G \mid g \cdot x = y\}$$

is a left coset of G_x.

(iii) Deduce that there is a bijection between the orbit $G \cdot x$ and the set G/G_x of left cosets of G_x.

[10 marks]

(b) Let D_3 be the standard dihedral subgroup of $\mathsf{O}(2)$ of order 6, generated by r_0 and $R_{2\pi/3}$. Let $f(x,y) = (x^2 - y^2,\ -2xy)$. Show f is a D_3-equivariant map of the plane to itself. [Hint: You may like to use complex numbers.]

[6 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, where $A \in \mathsf{O}(2)$ and $\mathbf{v} \in \mathbb{R}^2$, and deduce that multiplication in the group is given by

$$(A \mid \mathbf{v})(B \mid \mathbf{u}) = (AB \mid \mathbf{v} + A\mathbf{u}).$$

[4 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in 2\mathbb{Z},\ x + \tfrac{1}{2}y \in 2\mathbb{Z}\right\}.$$

(i) Show that the two vectors $\mathbf{u}_1 = (2,0)$ and $\mathbf{u}_2 = (0,4)$ both belong to L, and sketch a diagram showing all the points of L that lie in the rectangle $0 \leq x \leq 6$ and $0 \leq y \leq 8$. Deduce that $L \neq \mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case.

(iii) Show that the point group of the lattice L has order 4 by finding appropriate elements of its symmetry group $\mathscr{W}_L < \mathsf{E}(2)$, expressed in the form $(A \mid \mathbf{v})$, and state why it is Abelian.

[16 marks]

4. Consider a system of 3 coupled cells governed by the system of differential equations

$$\left\{\begin{array}{rcl} \dot{x} & = & x - y^2 \\ \dot{y} & = & y - z^2 \\ \dot{z} & = & z - x^2. \end{array}\right. \qquad (*)$$

(a) Show that the system $(*)$ has $\mathbb{Z}_3$ symmetry, and draw a cell diagram illustrating the system. Explain briefly why the system does not have S_3 symmetry. [6 marks]

(b) Deduce that the diagonal, where $x = y = z$, is invariant under the evolution of the system, stating carefully any results used. [4 marks]

(c) Find all the equilibrium points that lie on the diagonal. [4 marks]

(d) Find the unique solution to this system with initial value $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. [6 marks]

(e) Suppose $\gamma(t)$ is a periodic solution with fundamental period T of a differential equation with symmetry group G. Define the symmetry group of the periodic solution. If the system $(*)$ has any periodic orbits with non-trivial symmetry, show that the only possible symmetry types are $\mathbb{Z}_3$ and $\widetilde{\mathbb{Z}}_3$, making explicit what each means for the orbit. [10 marks]

END OF EXAMINATION PAPER