Two hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN GEOMETRY AND NATURE

16 May 2019

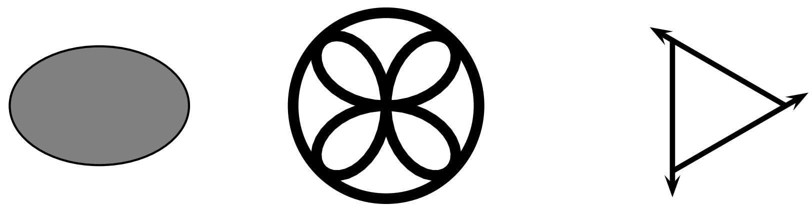
14.00 – 16.00

Answer **ALL FIVE** questions

Electronic calculators may be used, provided that they cannot store text.

1.

(a) Name the symmetry groups of the following three planar shapes, and state in each case the order of the group and whether the group is Abelian.

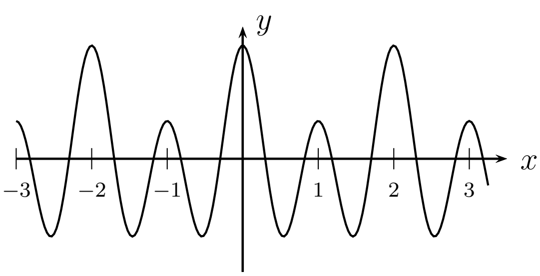


(i) (ii) (iii)

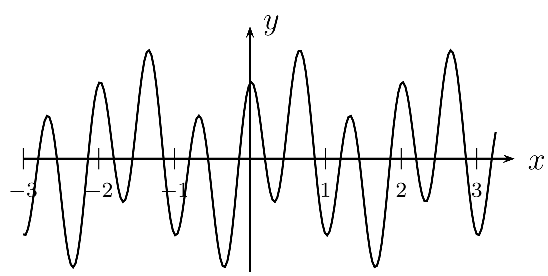
[6 marks]

(b) Given a function $f : \mathbb{R} \to \mathbb{R}$, its *extended symmetry group* consists of all pairs (T, σ) for which T is a transformation of $\mathbb{R}$ and $\sigma \in \mathbb{Z}_2 = \{\pm 1\}$ satisfying $f(Tx) = \sigma f(x)$ (in particular, $f(Tx) = \pm f(x)$).

Describe the extended symmetry group of each of the functions f and g of one variable whose graphs are shown below. (It is sufficient to give generators of each of the groups.)



$y = f(x)$



$y = g(x)$

[6 marks]

2. Suppose a finite group G acts on a set X.

(a) Define the orbit $G \cdot x$ and stabilizer G_x of an element $x \in X$. Show that the stabilizer is always a subgroup of G.

[5 marks]

(b) State the orbit-stabilizer theorem.

[3 marks]

(c) Let $h \in G$ and $y = h \cdot x$. Show that the subset

$$G_{x,y} = \{g \in G \mid g \cdot x = y\}$$

is a left coset of G_x, and explain briefly how this can be used to prove the orbit-stabilizer theorem (details of the proof are not required).

[6 marks]

(d) The 'octahedral group' $\mathbb{O}$ of order 24 acts by rotations on the cube. On each face of the cube the 2 diagonals are drawn. Use the orbit-stabilizer theorem to find the order of the stabilizer of one of these diagonals.

[4 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, where $A \in \mathsf{O}(2)$ and $\mathbf{v} \in \mathbb{R}^2$. Show that multiplication in the group is given by

$$(A \mid \mathbf{v})(B \mid \mathbf{u}) = (AB \mid \mathbf{v} + A\mathbf{u}),$$

and deduce the Seitz symbol of the inverse transformation $(A \mid \mathbf{v})^{-1}$.

[6 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in \mathbb{Z},\ x + y \in 3\mathbb{Z}\right\}.$$

(i) Show that the two vectors $\mathbf{u}_1 = (3, 0)$ and $\mathbf{u}_2 = (0, 3)$ both belong to L. Sketch a diagram showing all the points of L that lie in the rectangle $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$ and find a point in L which is not contained in $\mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$, as required for the classification of symmetry types of lattices; that is, such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, and $|\mathbf{a}| \leq |\mathbf{b}| \leq |\mathbf{b} - \mathbf{a}| \leq |\mathbf{b} + \mathbf{a}|$, proving carefully that your choice satisfies this.

(iii) Identify which of the 5 types of lattice L is (Oblique, Rectangular, Centred rectangular, Square or Hexagonal).

[16 marks]

4. Consider a system of 3 coupled cells governed by the system of differential equations

$$\begin{cases} \dot{x} &= x - yz \\ \dot{y} &= y - xz \\ \dot{z} &= z - xy. \end{cases} \tag{$*$}$$

(a) Show that the system $(*)$ has S_3 symmetry, and draw a cell diagram illustrating the system.

[4 marks]

(b) Deduce that the diagonal, where $x = y = z$, is invariant under the evolution of the system, stating carefully any results used.

[4 marks]

(c) Find all the equilibrium points of the system. Divide them into group orbits and find the Burnside type of the set of equilibrium points.

[6 marks]

(d) Let $\mathbf{x}(t)$ be the solution to this system with initial value $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Find which of the equilibria is equal to $\lim_{t\to\infty} \mathbf{x}(t)$. [You are not expected to solve the differential equation.]

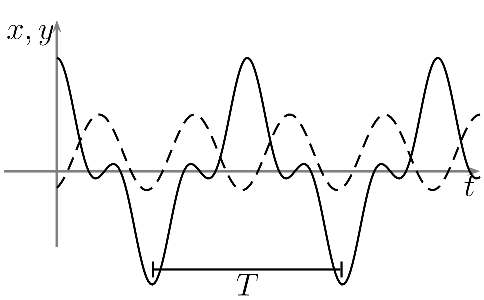
[4 marks]

5.

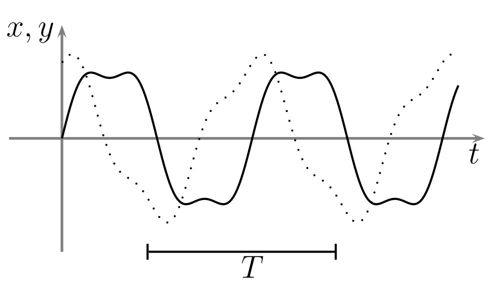
(a) Define the symmetry group of a periodic solution $\gamma(t)$ with fundamental period T of a differential equation with symmetry group G.

[3 marks]

(b) Suppose a mechanical system in the plane has potential energy, with D_2 symmetry; that is $V(x, y) = V(-x, y) = V(x, -y)$. Let γ be a periodic orbit with period T and symmetry group $\widetilde{\mathsf{C}_2}$, and let $\delta(t)$ be a periodic solution of the same period and with symmetry $\widetilde{D_1}$. Write down in each case what the symmetry group tells us about $x(t)$ and $y(t)$ for each periodic orbit. Below is shown numerical data taken from two experiments with this symmetry showing both $x(t)$ and $y(t)$. Determine which solution is $\gamma(t)$ and which is $\delta(t)$, giving a brief justification of your answer.



Experiment 1



Experiment 2

[7 marks]

END OF EXAMINATION PAPER