2 hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN NATURE

19 January 2018

09:45–11:45

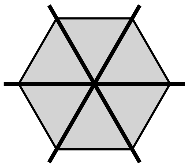
Answer **ALL FOUR** questions

Electronic calculators may be used, provided that they cannot store text, transmit or receive information, or display graphics

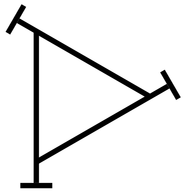
---

1.

(a) Name the symmetry groups of the following two planar shapes, and state in each case the order of the group and whether the group is Abelian.



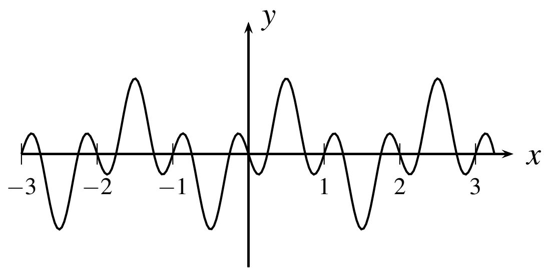
(i)



(ii)

[4 marks]

(b) Consider the periodic function f of one variable whose graph is shown below.



$y = f(x)$

By inspecting the graph, find the symmetry group of the function f allowing for possible change in sign of the function. More precisely, consider the action of the group $\mathsf{E}(1) \times \mathbb{Z}_2$, where $\mathsf{E}(1)$ is the Euclidean group in 1 dimension, and $(\phi, s) \in \mathsf{E}(1) \times \mathbb{Z}_2$ acts on a function by

$$(\phi, s) \cdot f = sf \circ \phi^{-1}.$$

Here $s \in \mathbb{Z}_2 = \{1, -1\}$ with multiplication, and is acting by changing the sign of f.

Verify that the function $f(x) = \sin(\pi x) - \sin(3\pi x)$ has the symmetry you found.

You may assume that every transformation $T \in \mathsf{E}(1)$ is either a translation of the form $T(x) = u + x$ or a reflection of the form $T(x) = u - x$ for some $u \in \mathbb{R}$. [8 marks]

2. Consider the subgroups C_4 and D_4 of $\mathsf{O}(2)$.

(a) Write down the orbit of the point $p = (1,1) \in \mathbb{R}^2$ under each of the C_4 and D_4 actions. State the orbit-stabilizer theorem and explain how these orbits illustrate this. [8 marks]

(b) Let X be the set of 12 quadratic functions on the plane,

$$X = \left\{ax^2 + bxy + cy^2 \mid a, c \in \{-1,1\},\ b \in \{-1,0,1\}\right\},$$

(for example $f_1(x,y) = x^2 - xy + y^2$ and $f_2(x,y) = x^2 - y^2$ are both elements of X). Let D_4 act on X by $g \cdot f = f \circ g^{-1}$, where $f \in X, g \in \mathsf{D}_4$.
(i) Show that this action is not effective.
(ii) Find those functions that are fixed by this action of D_4.
(iii) Determine the Burnside type of this action. [16 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, where $A \in \mathsf{O}(2)$ and $\mathbf{v} \in \mathbb{R}^2$, and deduce that multiplication in the group is given by

$$(A \mid \mathbf{v})(B \mid \mathbf{u}) = (AB \mid \mathbf{v} + A\mathbf{u}).$$

Use this to find the Seitz symbol for the inverse of the element $(A \mid \mathbf{v})$. [6 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x,y) \in \mathbb{R}^2 \mid x+y \in 2\mathbb{Z},\ x-y \in 2\mathbb{Z}\right\}.$$

(i) Show that $(x,y) \in L$ implies $x, y \in \mathbb{Z}$, and find an example of integers x, y for which $(x,y) \notin L$. Sketch a diagram showing all the points of L that lie in the square $-2 \leq x \leq 4$ and $-2 \leq y \leq 4$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case. Name the symmetry type of the lattice (one of 'Oblique', 'Rectangular', 'Centred rectangular', 'Square' or 'Triangular'), giving a brief explanation.

(iii) Define a glide reflection in the plane. Find a glide reflection $(A \mid \mathbf{v})$ in $\mathscr{W}_L$ whose line of reflection is not the line of reflection of a reflection symmetry of L.

[18 marks]

4. Consider the following system of ordinary differential equations,

$$\begin{cases} \dot{x} &= -x+yz^2 \\ \dot{y} &= -2y+xz^2 \\ \dot{z} &= z(1-x^2-y^2-z^2). \end{cases} \qquad (*)$$

Consider the action of the group $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Show that the system $(*)$ has symmetry G. Deduce that the x-y plane and the z-axis are each invariant under the evolution of the system, stating carefully any results used. [10 marks]

(b) Find all the equilibrium points that lie on these subspaces. [5 marks]

(c) Find the unique solution to this system with initial value $(x,y,z) = (1,1,0)$. What is the limit as $t \to \infty$ of this solution? [5 marks]

END OF EXAMINATION PAPER