Feedback on the MATH35082 coursework of March 2020

I marked this taking into account the presentation of the arguments. Sloppiness in the explanation loses marks.

Question 1

(a) [2 marks] Almost everyone did this well

(b) [4 marks] Most did this well. A few decomposed $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\|$ but for example had $\mathbf{v}_\| = (3, 0)$ and $\mathbf{v}_\perp = (0, 1)$. But these are not parallel nor perpendicular to the line of reflection. Some omitted to draw a diagram, which was an unfortunate mistake (but marks can't be given if the work isn't there!).

Question 2 (a) [3 marks] Generally done well. Two of the sloppy statements were,

• *We want to show* $\exists m, n \in \mathbb{Z}$ *such that* $\mathbb{Z}^2 = \{m\mathbf{a} + n\mathbf{b} \mid m, n, \in \mathbb{Z}\}$. This just doesn't make any sense! Saying, *we want to show that* $\mathbb{Z}^2 = \{m\mathbf{a} + n\mathbf{b} \mid m, n, \in \mathbb{Z}\}$, would have been fine.

• *We want to show* $\exists m, n \in \mathbb{Z}$ *such that* $m\mathbf{a} + n\mathbf{b} = (1, 0)$ *and* $m\mathbf{a} + n\mathbf{b} = (0, 1)$. It's not possible for both to be satisfied for the same values of m and n!

(b) [4 marks] This was also mostly done well. A few scripts claimed to be proving $A \Rightarrow B$ and then gave an argument showing $B \Rightarrow A$; this confusion lost a mark.

A fairly common mistake in proving $A^{-1} \in \mathrm{Mat}_2(\mathbb{Z}) \Rightarrow \det A = \pm 1$ was to write if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ (which is correct), but then to conclude with no further argument that for the entries of A^{-1} to be integers, we must have $ad - bc = \pm 1$. While this is true, it needs some argument: to have, eg, $\frac{d}{ad-bc} \in \mathbb{Z}$ you don't need $ad - bc = \pm 1$, just that $ad - bc$ divides d. Those (few) who did give the argument, well done! A few wrote the explicit statement,

$$p, q \in \mathbb{Z},\ \frac{p}{q} \in \mathbb{Z} \quad \Rightarrow \quad q = \pm 1.$$

I'll leave it to you to find a counter-example!

The easier proof that many gave is to note that if $A, A^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$ then $\det(A), \det(A^{-1}) \in \mathbb{Z}$. But since $\det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$ then $\det A$ divides 1 so is equal to ± 1.

(c) [3 marks] Mostly done well. Some used part (b), some didn't. Either approach was acceptable. One error was to write that (a, b) and (c, d) are linearly independent and therefore generate $\mathbb{Z}^2$. But eg $(2, 0)$ and $(0, 2)$ are linearly independent but don't generate $\mathbb{Z}^2$ (using integer coefficients).

Question 3 [4 marks] A reasonable discussion about duality gave 3 out 4 marks. The crucial point is that since the vertices of the octahedron lie at the *centre* of the faces of the cube. Then, since any symmetry of the cube takes faces to faces, it therefore takes the centre of a face to the centre of a face, thereby permuting the vertices of the octahedron. And vice versa. Anything that reflected this point gained the extra mark. Alternatively (although slightly less precise), pointing out that the two dual polyhedra have the same axes and planes of symmetry will gain that extra mark. Better was to show the common axes and planes of symmetry in the diagrams. [One or two people did this very well.]

Note for anyone who's done Algebraic Structures 2, or Commutative Algebra: Question 2(b) remains true if $\mathbb{Z}$ is replaced by any commutative ring with unit R: a matrix $A \in \mathrm{Mat}_n(R)$ is invertible (in $\mathrm{Mat}_n(R)$) iff its determinant is a unit in R. And the proof is essentially the same. Note that commutativity is essential, otherwise the determinant is not defined.