Chapter 6

Symmetries of periodic motion

In an ODE, a ***periodic orbit*** is a (non-constant) solution $\gamma(t)$ for which there is $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. In other words, $\gamma(t)$ repeats itself periodically. If T satisfies this, so does any kT for $k \in \mathbb{Z}$. For instance, $\gamma(t+2T) = \gamma(t+3T) = \gamma(t-T) = \gamma(t)$. The least $T > 0$ satisfying this is called the ***fundamental period***, which we write as T_0. Note that if $\gamma(t)$ is a periodic orbit, then $\gamma(t)$ cannot be an equilibrium solution as by definition $\gamma(t)$ is not constant.

As a periodic motion progresses, the *phase* refers to the proportion of the period covered. For example, the Earth moves around the Sun in a periodic motion with fundamental period $T_0 = 1$year, and the difference between the positions on Jan 1st and April 1st would be a change of phase of $^1/_4$.

More precisely, if a periodic orbit γ has fundamental period T_0, then a ***change of phase*** of θ is the motion from $\gamma(t)$ to $\gamma(t+\theta T_0)$ (for any t).

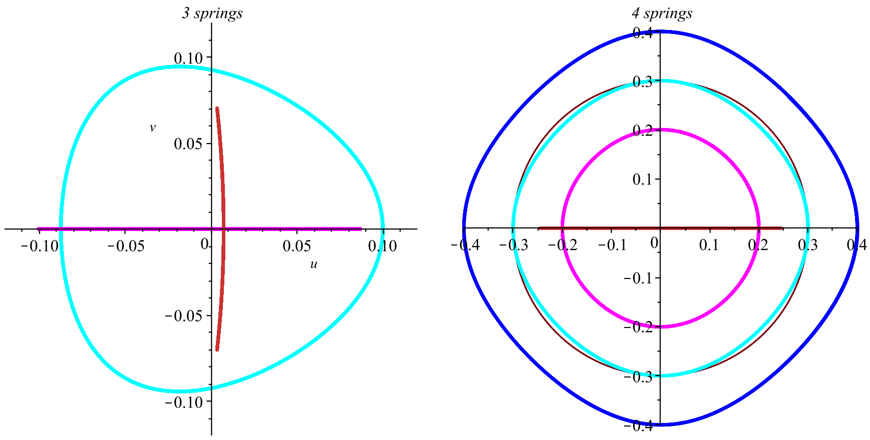
The ***circle group*** is defined to be the group $S^1 \simeq \mathbb{R}/\mathbb{Z}$ (see the appendix). In practice, we identify

$$S^1 = \{\theta \in [0,1] \mid \theta = 0 \text{ and } \theta = 1 \text{ are identified}\},$$

but all calculations are modulo $\mathbb{Z}$. For example $\frac{3}{4} + \frac{3}{4} = \frac{3}{2} \equiv \frac{1}{2}$. Note that the phase as described above can be viewed as an element of S^1 (we'll see this in more detail below).

6.1 Spatio-temporal symmetry

Consider the outer (dark blue) periodic orbit in Fig. 6.1(b). This curve forms a square with rounded corners, and looks like it has D_4 symmetry (it does!). However, the curve represents a motion, and it would be going either clockwise around the origin or anticlockwise. The reflections in D_4 will change the direction of motion, and we don't want to count that as a symmetry, and that leaves the 4 rotations of D_4. Thus the symmetry group is C_4. However, we want to include in the symmetry the change in time (or phase) of the motion, given by each element. So for example, if we consider γ to



(a) (b)

FIGURE 6.1: These are numerical plots showing symmetric periodic solutions to (a) the 3-spring ODE and (b) the 4-spring ODE, both described in Chapter 4. In (a) the horizontal periodic orbit has symmetry D_1, the almost vertical one $\widetilde{\mathrm{D}_1}$ while the almost circular one has symmetry $\widetilde{\mathrm{C}_3}$. See also Example 6.10 below for more details.

be the anticlockwise solution along that curve, then the rotation $R_{\pi/2}$ involves a time shift by one quarter of a period. This idea gives rise to the so-called 'spatio-temporal' symmetry group of a periodic orbit. This notion of the spatio-temporal symmetry group of a periodic orbit was introduced in an important paper [7] by Martin Golubitsky and Ian Stewart published in 1985.

Definition 6.1. Let $\gamma(t)$ be a periodic orbit, with fundamental period $T_0 > 0$, of an ODE with symmetry group G. The ***spatio-temporal symmetry group*** Σ_γ of γ is the group

$$\Sigma_\gamma = \{(g,\theta) \in G \times S^1 \mid g\cdot\gamma(t) = \gamma(t+\theta T_0),\ \forall t\}.$$

It is a subgroup of the Cartesian product of G with S^1. ✫

Thus (g,θ) lies in the symmetry group of a periodic orbit if the action of g on the orbit coincides with a change of phase by θ.

Exercise: Check Σ_γ is indeed a subgroup of $G\times S^1$ (you can show this directly, but see Problem 6.7 for a different argument).

Example 6.2. (Spring systems: see Fig. 6.1) The periodic orbit $\gamma(t)$ shown in light blue in Figure 6.1(a), where the motion is anticlockwise, has spatio-temporal symmetry group equal to the subgroup of $\mathrm{D}_3 \times S^1$,

$$\Sigma_\gamma = \{(I,0),(R_{2\pi/3},{}^1\!/_3),(R_{4\pi/3},{}^2\!/_3)\}. \tag{6.1}$$

For example, the element $(R_{2\pi/3},{}^1\!/_3)$ means that rotating the plane by $2\pi/3$ advances the orbit by 1/3 of a period. This symmetry group is denoted $\widetilde{C}_3$, where the tilde informs us of the time component in the symmetry group. The dark pink (magenta) solution lies on the x-axis and has symmetry group

$$\Sigma_\gamma = \{(e,0),\ (r_0,0)\}.$$

This means that $r_0 \cdot \gamma(t) = \gamma(t)$, that is $\gamma(t)$ lies in the line $\mathrm{Fix}(r_0)$ (the line of reflection) for all t. In this case there is no 'change of phase' symmetry, and this symmetry group is purely spatial: $\Sigma_\gamma = \mathsf{D}_1$.

Similarly, the nearly circular orbits in (b) have symmetry group

$$\Sigma_\gamma = \{(I,0),(R_{\pi/2},{}^1\!/_4),(R_\pi,{}^1\!/_2),(R_{3\pi/2},{}^3\!/_4)\}.$$

These can be checked on the figures. This symmetry group is denoted $\widetilde{C}_4$. (Again, the tilde refers to the time component.)

Note that in both examples, the orbits are rotating anticlockwise. The reflection in the x-axis is a symmetry of the set of points in the orbit, but not of the orbit itself because it reverses the direction of rotation (so doesn't satisfy the definition of symmetry). There are similar orbits rotating in the opposite direction. For example, there is one with the symmetry group,

$$\Sigma_\gamma = \{(I,0),(R_{2\pi/3},{}^2\!/_3),(R_{4\pi/3},{}^1\!/_3)\},$$

to be contrasted with (6.1). (Note that ${}^2\!/_3 \equiv {}^{-1}\!/_3$ in S^1.) This symmetry group is also denoted $\widetilde{C}_3$.

The question remains, given a system with symmetry group G, which subgroups of $G \times S^1$ can arise as symmetry groups of periodic orbits? The following result helps us to answer to that question.

Proposition 6.3. *Let $\gamma(t)$ be a periodic orbit with fundamental period $T_0 > 0$ and with symmetry group $\Sigma_\gamma \le G \times S^1$. Consider the projection $\pi : \Sigma_\gamma \to G$ defined by $\pi(g,\theta) = g$. Then, π is an isomorphism onto its image.*

Definition 6.4. A subgroup Σ of $G \times S^1$ is said to be a ***spatio-temporal*** subgroup if the projection π is an isomorphism of Σ with its image in G. ✯

The proposition above says that the spatio-temporal symmetry of a periodic orbit is a spatio-temporal subgroup of $G \times S^1$!

For the symmetry group in (6.1), we have

$$\pi(\Sigma_\gamma) = \{I, R_{2\pi/3}, R_{4\pi/3}\} = \mathsf{C}_3 \le \mathsf{D}_3.$$

We can see that $\pi : \Sigma_\gamma \to \mathsf{C}_3$ is an isomorphism so that this Σ_γ is indeed a spatio-temporal subgroup.

Proof: To prove the proposition, we first check that π is a homomorphism:

$$\begin{aligned} \pi((g,\theta)(h,\phi)) &= \pi(gh,\theta+\phi) \\ &= gh \\ &= \pi(g,\theta)\pi(h,\phi). \end{aligned}$$

It is clearly surjective, by definition of the image. To show injectivity, observe that

$$\begin{aligned} \ker(\pi) &= \{(g,\theta)\in\Sigma_\gamma \mid g=e\} && (6.2) \\ &= \{(e,\theta)\in\Sigma_\gamma\} && (6.3) \end{aligned}$$

In particular, $(e,\theta)\in\Sigma_\gamma \implies \gamma(t+\theta T_0) = e\cdot\gamma(t) = \gamma(t)$. But $0\le\theta<1$. If $\theta>0$, then $\gamma(t)$ has period $\theta T_0 < T_0$ which is in contradiction with the minimality of the fundamental period T_0. Therefore, $\theta=0$ and π is injective. Thus π is an isomorphism, as stated. ❐

As a consequence, let $H_\gamma := \pi(\Sigma_\gamma)$ which is a subgroup of G. The proposition tells us that H_γ and Σ_γ are isomorphic, and that π provides the isomorphism. Thus for each $h\in H_\gamma$, there is a unique $\theta=\theta(h)\in S^1$ such that $(h,\theta(h))\in\Sigma_\gamma$. That is, $\theta: H_\gamma\to S^1$ is a map, whose graph is precisely the subgroup Σ_γ. It follows (see Exercise A3.5 in the appendix) that θ is a homomorphism. (This is also easy to show directly.)

Conclusion: Given an ODE with symmetry G, the possible spatio-temporal symmetry groups of periodic orbits are given by two pieces of data:

- a subgroup $H\le G$
- a homomorphism $\theta: H\to S^1$.

We may therefore denote the symmetry group of a periodic orbit as a pair (H,θ), where H is a subgroup of G and $\theta: H\to S^1$ is a homomorphism, and then the symmetry group is the graph of θ.

Since a group may have many subgroups, providing a list of all possible symmetry types may be a lengthy procedure, but it is not difficult. If two subgroups are conjugate, then it is enough to study one of them. This is because, if (H,θ) is a spatio-temporal subgroup of $G\times S^1$ then so is (H',θ') where $H' = gHg^{-1}$ and $\theta'(h') = \theta(g^{-1}h'g)$ (for $h'\in H'$).

Example 6.5. Consider $G=\mathsf{D}_3$ acting on V ($=\mathbb{R}^2$ for instance, as in the 3-springs system). We want to find the possible spatio-temporal subgroups of $\mathsf{D}_3\times S^1$.

First we list the subgroups of D_3. There are 4 types of subgroup (that is, any subgroup is conjugate to one of these):

- $\mathbb{1}$
- $\mathsf{D}_1 = \langle r_0\rangle$ and two conjugate copies of this: $\langle r_{\pi/3}\rangle$ and $\langle r_{-\pi/3}\rangle$.
- $\mathsf{C}_3 = \langle R_{2\pi/3}\rangle$

- D_3

For each one of these, what homomorphisms $\theta : H \to S^1$ are there?

- For $H = \mathbb{1}$, the only possible homomorphism $\theta : \mathbb{1} \to S^1$ is $\theta(I) = 0$ as a homomorphism maps the identity element to the identity element. Thus $(H,\theta) = (\mathbb{1}, 0)$.

- $H = \mathsf{D}_1$, $\theta : \mathsf{D}_1 \to S^1$. Since it is a homomorphism, $\theta(r_0^2) = \theta(r_0) + \theta(r_0) = 2\theta(r_0) \bmod \mathbb{Z}$ in $\mathbb{Z}/\mathbb{R}$. But $r_0^2 = I$ and $\theta(I) = 0$. Therefore, $2\theta(r_0) = 0 \bmod \mathbb{Z}$. Thus, $\theta(r_0) = 0$ or $\theta(r_0) = 1/2$

  i) If $\theta(r_0) = 0$, then $\Sigma_\gamma = \{(I, 0), (r_0, 0)\}$. This says $\gamma(t + 0I) = r_0 \cdot \gamma(t)$ i.e. $\gamma(t) = r_0 \cdot \gamma(t)$ for all $t \in \mathbb{R}$; that is, $\gamma(t) \in \mathrm{Fix}(r_0, V)$ for all t. We call this subgroup simply D_1 (even though it is really $\mathsf{D}_1 \times \{0\} < \mathsf{D}_3 \times S^1$).

  ii) If $\theta(r_0) = 1/2$, then $\Sigma_\gamma = \{(I, 0), (r_0, 1/2)\}$. This says $\gamma(t + 1/2T) = r_0\gamma(t)$. This is denoted $\widetilde{\mathsf{D}_1}$, the tilde over the top telling us that there is a time component to the symmetry.

  The two possibilities with $H = \mathsf{D}_1$ are therefore $(\mathsf{D}_1, 0)$ and (D_1, θ_1), where $\theta_1(r_0) = 1/2$.

- For $H = \mathbb{Z}_3 = \langle R_{2\pi/3} \rangle$, let us find the homomorphisms $\theta : \mathbb{Z}_3 \to S^1$. Now, $\theta(R^3_{2\pi/3}) = 3\theta(R_{2\pi/3}) \bmod \mathbb{Z}$. But $\mathbb{R}^3_{2\pi/3} = I$ and thus $3\theta(R_{2\pi/3}) = 0 \bmod \mathbb{Z}$. Possibilities are $\theta(R_{2\pi/3}) = 0, \frac{1}{3}, \frac{2}{3}$. Analogous to the previous case (where $H = \mathbb{Z}_2$), if $\theta(R_{2\pi/3}) = 0$, then $\gamma(t) \in \mathrm{Fix}(R_{2\pi/3}) = \mathrm{Fix}(\mathbb{Z}_3)$ and this is denoted $\mathbb{Z}_3$.

  If $\theta(R_{2\pi/3}) = \frac{1}{3}$, then

  $$\Sigma_\gamma = \{(I, 0), (R_{2\pi/3}, {}^1\!/_3), (R_{4\pi/3}, {}^2\!/_3)\}.$$

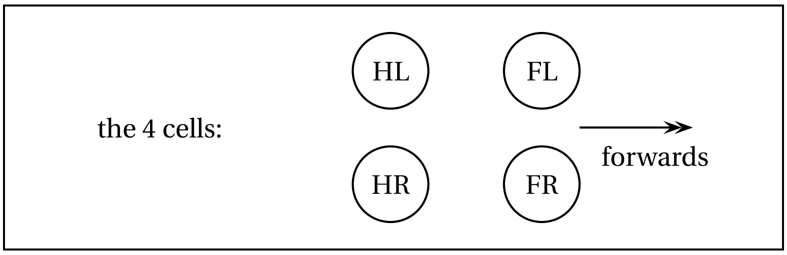
  This says $\gamma(t + \frac{1}{3}T) = R_{2\pi/3}\gamma(t)$. This is denoted $\widetilde{\mathsf{C}_3}$ (because of the time component). The final possibility $\theta(R_{2\pi/3}) = {}^2\!/_3$ is similar, and is also denoted $\widetilde{\mathsf{C}_3}$.

- Finally, for $H = \mathsf{D}_3$, there are the following homomorphisms:
  - $\theta : \mathsf{D}_3 \to \{0\}$ the trivial homomorphism,
  - $\theta(r_0) = 1/2$ and $\theta(R_{2\pi/3}) = 0$. NB: $\ker\theta$ is a normal subgroup of D_3 (and we have seen that $\mathsf{C}_3 \lhd \mathsf{D}_3$ is the only non-trivial one).

6.2 Animal gaits

Animal gaits provide an interesting example of periodic motion with symmetry, for a system treated as coupled cells. We discuss briefly one example here, while further information can be found in Chapter 3 of the book by Golubitsky and Stewart [6], and in papers cited in that book.

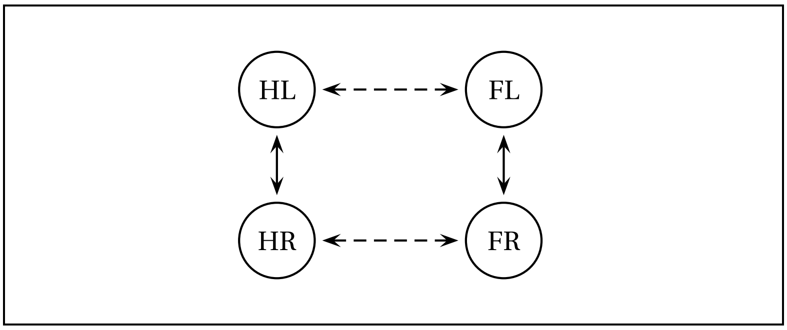
Imagine a 4-legged animal (horse, dog, elephant, camel ...) and model each of the 4 legs, or rather its control mechanism, as a single cell (which would be the first level of modelling: rather naïve, but let's see how far we get). Label these cells FL, FR, HL, and HR (FL = Front Left and HR = Hind Right, etc.). Biologically, each cell could represent a group of neurons whose 'firing' controls the movement of the leg.



The question is, how should these be coupled? One would expect the coupling between the right legs to be the same as between the left legs, and that coupling from right to left is the same as from left to right. This gives a left-right $\mathbb{Z}_2$ symmetry.

However, any further assumptions on the coupling, and consequent symmetries, would be more contentious. Would there be a front-back symmetry? The fact that quadrupeds prefer to walk forwards may be due to a lack of symmetry in the coupling, but it also might be due instead to the shape of legs and knees. However, a rotational symmetry by $\pi/2$ seems less likely: there would probably be a different coupling between FR and FL than between FR and HR.

Let us therefore suppose a coupling with symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathsf{D}_2$:



Question: what are the possible symmetries of periodic motions for a system with this symmetry?

Answer: The group is D_2. Let us call the generators σ_1, σ_2, where, as permutations of the cells,

$$\sigma_1 = \begin{pmatrix} FL & FR & HL & HR \\ FR & FL & HR & HL \end{pmatrix}, \quad \text{and} \quad \sigma_2 = \begin{pmatrix} FL & FR & HL & HR \\ HL & HR & FL & FR \end{pmatrix}.$$

That is, σ_1 swaps right and left, and σ_2 swaps front and hind legs. The product $\sigma_1\sigma_2$ 'rotates the animal' by π.

There are 11 spatio-temporal subgroups of $\mathsf{D}_2 \times S^1$ (see Problem 6.3) but only some of these are seen as gaits:

Trot When a horse is trotting, the diagonal pairs of legs hit the ground in unison: (FL+HR), (FR+HL) each pair half a period out of phase. This corresponds to the symmetry group

$$\Sigma_{\text{trot}} = \langle(\sigma_1, {}^1/_2),\ (\sigma_2, {}^1/_2)\rangle\,.$$

Thus $(\sigma_1\sigma_2, 0)$ is a symmetry, telling us that FL and HR move together, as do FR and HL.

Pace This is a common gait for camels, and occurs when the legs on the same side move in unison, (FL+HL), (FR+HR). The symmetry of this is

$$\Sigma_{\text{pace}} = \langle(\sigma_1, {}^1/_2),\ (\sigma_2, 0)\rangle\,.$$

Bound Here the front legs move in unison, and the hind legs in unison: (FR+FL), (HR+HL). The symmetry group is

$$\Sigma_{\text{bound}} = \langle(\sigma_1, 0),\ (\sigma_2, {}^1/_2)\rangle\,.$$

Pronk Here all four legs move in unison (a jump), and is sometimes seen in young lambs or deer. The spatio-temporal symmetry group (if the pronking is periodic) is therefore purely spatial, with $\Sigma_{\text{pronk}} = \mathsf{D}_2$.

Gallop Here we take the sequence to be HL, HR, FL, FR. This has symmetry

$$\Sigma_{\text{gall}} = \langle(\sigma_2, {}^1/_2)\rangle\,.$$

This symmetry group does not determine the sequence of motions, as it doesn't tell us for example, that HR comes mid-way between HL and FL, it could be at any time in the cycle.

To gain more information about the gallop would require a symmetry group with an element of order 4 coupled with a phase change of ${}^1/_4$. Moreover, in reality a gallop is more complex than indicated, as there is a 'pause' or 'suspension' between the FR and HL giving a total period of more than 4 'beats' (where a beat is the time interval between successive legs hitting the ground). One approach is to allow more cells in the model, see [6, Sec. 3.5] for a model with 8 cells.

6.3 Existence of symmetric periodic orbits in mechanical systems

In the previous section we described how to find all possible symmetries of periodic orbits of symmetric systems, but with no thought to whether they actually exist. We end with a result that guarantees the existence of periodic orbits with certain symmetries. This is the 'simplest possible' theorem in this direction; more general statements can be obtained with weaker hypotheses but require more detailed knowledge of the specific system.

In particular, this approach requires that W is an ***irreducible*** representation of G: First we need to define a mechanical system.

Definition 6.6. A mechanical system on W consists of two functions, ***kinetic energy*** $K : W \to \mathbb{R}$ which is a positive definite quadratic form on W, and ***potential energy*** which is a smooth function $V : W \to \mathbb{R}$. If W is a representation of a group G then the mechanical system is said to have ***symmetry*** G provided both K and V are invariant. ✯

Given the kinetic and potential energies, the equation of motion is given by what is essentially Newton's second law,

$$\mathbb{K}\ddot{\mathbf{x}} = -\nabla V(\mathbf{x}), \tag{6.4}$$

where $\mathbb{K}$ is the (invertible) symmetric matrix of the kinetic energy quadratic form: $K(x) = \frac{1}{2}\mathbf{x}^T\mathbb{K}\mathbf{x}$ (we won't be using this again). With this notation, it is easy to check that K is G-invariant if and only if $g\mathbb{K} = \mathbb{K}g$ (for all $g \in G$).

Definition 6.7. A representation W of G is ***irreducible*** if the only subspaces W' of W which are invariant under G, are $W' = \{0\}$ and $W' = W$. ✯

The importance of being irreducible is that, if V is an invariant potential energy, then (i) $W^G = \{0\}$ so the origin is a critical point of V, and (ii) both the Hessian matrix of V at the origin and the kinetic energy matrix $\mathbb{K}$ are scalar multiples of the identity. We will need to assume the Hessian of V is a *positive* multiple of the identity (which implies that the origin is a *stable* equilibrium).

Consider now the ***complexification*** of W, written $W^{\mathbb{C}}$. If $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a basis for W, then elements of W are linear combinations of these vectors with real coefficients. The complex vector space $W^{\mathbb{C}}$ is given by taking linear combinations of the same basis vectors, but now allowing complex coefficients:

$$\begin{aligned} W &= \left\{\sum_{j=1}^n x_j\mathbf{e}_j \mid x_j \in \mathbb{R}\right\}, \\ W^{\mathbb{C}} &= \left\{\sum_{j=1}^n z_j\mathbf{e}_j \mid z_j \in \mathbb{C}\right\}. \end{aligned}$$

For example complexifying $W = \mathbb{R}^n$ yields $W^{\mathbb{C}} = \mathbb{C}^n$.

Now G acts on W in a way where all the elements are represented by $n \times n$ matrices. One can therefore define an action of G on $W^{\mathbb{C}}$ by using the same matrices. For example, $R_\phi \in \mathsf{SO}(2)$ acts on $\mathbb{C}^2$ by

$$R_\phi \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\phi\, z_1 - \sin\phi\, z_2 \\ \sin\phi\, z_1 + \cos\phi\, z_2 \end{pmatrix}$$

Moreover (and this is the reason for using complex numbers), we can define an action of S^1 on $\mathbb{C}^n$ by $\theta \cdot \mathbf{z} = e^{2\pi i\theta}\mathbf{z}$ (scalar multiplication). That is,

$$\theta \cdot \left(\sum_j z_j\mathbf{e}_j\right) = \sum_j (e^{2\pi i\theta} z_j)\mathbf{e}_j.$$

For example for $n = 2$,

$$\theta \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{2\pi i\theta} z_1 \\ e^{2\pi i\theta} z_2 \end{pmatrix}.$$

Combining this with the given action of G on W and hence on $W^{\mathbb{C}}$ we obtain an action of $G \times S^1$ on $W^{\mathbb{C}}$ (since S^1 acts on $W^{\mathbb{C}}$ by multiplication by scalars, it commutes with the action of G, which is linear). Specifically,

$$(g,\theta)\cdot\mathbf{v} = \mathrm{e}^{2\pi\mathrm{i}\theta} g\,\mathbf{v}. \tag{6.5}$$

Moreover, an argument analogous to the proof of Proposition 4.10 shows that the fixed point space of any subgroup $\Sigma < G \times S^1$ is a complex subspace of $W^{\mathbb{C}}$.

Before stating the main theorem, let us examine how to find fixed point subspaces for the $G \times S^1$ action. Let $(g,\theta) \in G \times S^1$, and let $\mathbf{v} \in W^{\mathbb{C}}$ be a non-zero vector. Then, by Eq. (6.5), $\mathbf{v}$ is fixed by (g,θ) if $\mathrm{e}^{2\pi\mathrm{i}\theta} g\mathbf{v} = \mathbf{v}$. This can be written

$$g\mathbf{v} = \mathrm{e}^{-2\pi\mathrm{i}\theta}\mathbf{v}. \tag{6.6}$$

That is, $\mathbf{v}$ *is fixed by* (g,θ) *if and only if* $\mathbf{v}$ *is an eigenvector of* g *with eigenvalue* $\mathrm{e}^{-2\pi\mathrm{i}\theta}$.

Thus, $\mathbf{v} \in \mathrm{Fix}(\Sigma, W^{\mathbb{C}})$ if for each $g \in H = \pi(\Sigma)$ there is a $\theta = \theta(g)$ (ie, depending on g) such that $\mathbf{v}$ is an eigenvector of g with eigenvalue $\mathrm{e}^{-2\pi\mathrm{i}\theta}$. In particular $\mathbf{v}$ must be an eigenvector of g for every $g \in H$.

In this way, finding fixed point spaces is equivalent to finding eigenvectors of matrices. We will see this in action in the example below.

Definition 6.8. A subgroup $\Sigma \le G \times S^1$ is *complex-axial* if $\dim_{\mathbb{C}} \mathrm{Fix}(\Sigma, W^{\mathbb{C}}) = 1$. ✯

Theorem 6.9. *With this set up (assuming W to be irreducible and the Hessian of the potential to be positive definite), if $\Sigma \le G \times S^1$ is complex-axial, then in every neighbourhood of the origin, there are periodic orbits γ with symmetry group $\Sigma_\gamma = \Sigma$.*

We omit the proof of this theorem: it follows from the main existence theorem in [11], see also the book [6].

Example 6.10. We can use this theorem to prove the existence of periodic orbits with symmetries $\mathbb{Z}_2, \widetilde{\mathbb{Z}_2}, \widetilde{\mathsf{C}_3}$ in the 3-spring problem, see Figure 6.1(a), provided the origin is a stable equilibrium point (positive definite Hessian of the potential). Here we show how the $\widetilde{\mathsf{C}_3}$ and $\widetilde{\mathbb{Z}}_2$ arise (the other is left to the reader but doesn't involve any eigenvalues).

First consider the subgroup C_3 of D_3, which is generated by $R_{2\pi/3}$. The matrix of this generator is

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

The eigenvalues of this matrix are found to be $\mathrm{e}^{2\pi\mathrm{i}/3}$ and $\mathrm{e}^{-2\pi\mathrm{i}/3}$ (or equivalently, $-\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$). An eigenvector of $R_{2\pi/3}$ with eigenvalue $\mathrm{e}^{2\pi\mathrm{i}/3}$ is $(1, -\mathrm{i})^T$, and for the eigenvalue $e^{-2\pi i/3}$ one is $(1, i)^T$.

This shows us that $(R_{2\pi/3}, {}^1\!/_3)\cdot \mathbf{v} = \mathbf{v}$, for $\mathbf{v} = \begin{pmatrix}1\\ i\end{pmatrix}$, or indeed for $\mathbf{v} = \begin{pmatrix}\lambda\\ \lambda i\end{pmatrix}$, for any $\lambda \in \mathbb{C}$. Consequently

$$\mathrm{Fix}(R_{2\pi/3}, {}^1\!/_3) = \left\{(\lambda, \lambda i)^T \mid \lambda \in \mathbb{C}\right\}$$

which is a 1-dimensional subspace of $\mathbb{C}^2$, showing that the subgroup $\widetilde{\mathsf{C}_3}$ is complex-axial. Using the other eigenvalue, we see that

$$\mathrm{Fix}(R_{2\pi/3}, -{}^1\!/_3) = \left\{(\lambda, -\lambda i)^T \mid \lambda \in \mathbb{C}\right\}$$

showing that this group $\widetilde{\mathsf{C}'_3}$ is also complex-axial.

Now consider the subgroup $\mathbb{Z}_2 = \mathsf{D}_1$ of D_3 which is generated by r_0. The matrix of r_0 is $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ whose eigenvalues are ± 1. The $+1$ eigenvalue has eigenvector $(1, 0)^T$ and the -1 eigenvalue has eigenvector $(0, 1)^T$. Both eigenspaces are 1-dimensional, and the $+1$ eigenspace is fixed by $(r_0, 0) \in \mathsf{D}_3 \times \S^1$, generating the subgroup D_1, while the -1 eigenspace gives rise to a complex-axial subgroup denoted $\widetilde{\mathsf{D}_1}$ which is generated by $(r_0, \frac{1}{2}) \in \mathsf{D}_3 \times S^1$. These two complex-axial subgroups are both of order two (their projections to D_3 are both equal to D_1).

Therefore we can apply the theorem above to conclude the existence of periodic orbits with symmetry $\widetilde{\mathsf{C}_3}$, D_1 and $\widetilde{\mathsf{D}_1}$, which are shown in Figure 6.1(a). (The difference between $\widetilde{\mathsf{C}_3}$ and $\widetilde{\mathsf{C}'_3}$ is that the first runs anticlockwise while the second runs clockwise.

In Figure 6.1(b), note that the almost circular periodic orbits have symmetry $\widetilde{\mathbb{Z}}_4$ (as stated on p. 6.2), while the horizontal brown orbit (on the x-axis) has symmetry $\widetilde{\mathsf{D}}_2$, where

$$\widetilde{\mathsf{D}}_2 = \{(I, 0), (r_0, 0), (r_{\pi/2}, {}^1\!/_2), (R_\pi, {}^1\!/_2)\}.$$

Recall that $\mathsf{D}_2 = \langle r_0, r_{\pi/2}\rangle = \langle r_0, R_\pi\rangle$. The fact that $(r_0, 0) \in \Sigma_\gamma$ implies that for all t, $r_0\gamma(t) = \gamma(t)$, which explains why $\gamma(t)$ lies on the x-axis.

6.4 Problems

6.1 Use the property of uniqueness of solutions of ODEs to show that if γ is a solution for which there is a $T > 0$ such that $\gamma(T) = \gamma(0)$ then $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$.

6.2 Consider a system of 3 identical coupled cells with symmetry S_3. Draw the cell diagram for such a system. Let $\gamma(t)$ be a periodic orbit with symmetry $\widetilde{\mathbb{Z}_2}$ generated by $((1\ 2), \frac{1}{2}) \in S_3 \times S^1$ and period T. State the relation between the 3 cells after half a period, and deduce the period of cell 3.

6.3 By considering the subgroups of $G = \mathsf{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and all possible homomorphisms from these to $S^1 = \mathbb{R}/\mathbb{Z}$, find all possible symmetry groups of periodic orbits in a system with D_2 symmetry. [Hint: there are 5 subgroups, and these have 1, 2, 2, 2 and 4 homomorphisms respectively giving 11 possible symmetry groups in all.]

6.4 Use the symmetry to justify the last sentence of Example 6.10.

6.5 Repeat Example 6.10, but for the D_4 action on $\mathbb{R}^2$, showing the existence of periodic orbits with symmetries shown in Figure 6.1(b).

6.6 Find all 27 complex-axial symmetry groups of the action of $\mathbb{T}_d$ on $\mathbb{R}^3$ described in Section 4.5.

6.7 Suppose G acts on $\mathbb{R}^n$ and let X denote the space of all continuous maps $\gamma : S^1 \to \mathbb{R}^n$. There is an action of $G \times S^1$ on this space: if $\gamma \in X$ then define $(g,\theta)\cdot\gamma$ to be the map,

$$((g,\theta)\cdot\gamma)(t) = g\cdot(\gamma(t-\theta)).$$

Verify that this defines an action, and show that the stabilizer of an element γ is precisely its symmetry group Σ_γ.

