Chapter 5

Symmetric systems of ODEs

A ***system of first order*** **ODEs** in n variables is an equation of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}) \text{ (autonomous) or } \dot{\mathbf{x}} = f(\mathbf{x}, t) \text{ (non-autonomous)}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, or $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ (respectively). We concentrate on autonomous systems, but most of the ideas carry across without change to non-autonomous ones. A ***solution*** is a map $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = f(\gamma(t))$, for some $\varepsilon > 0$. The general existence and uniqueness theorem states that, if f is smooth, then given any initial condition $x_0 \in \mathbb{R}^n$, there exists a unique smooth solution $\gamma(t)$ satisfying $\gamma(0) = x_0$.

Unfortunately, in general one can only guarantee local solutions. The only general setting where global solutions are guaranteed is where $f(\mathbf{x}, t)$ is *linear* in $\mathbf{x}$.

In this chapter we introduce some of the ideas used in studying symmetric systems of ODEs; further theory, and some applications, can be found in the book *The Symmetry Perspective* by M. Golubitsky & I. Stewart [6]. Historically, the systematic understanding of symmetry properties of solutions of differential equations using group actions began in the early 1980s, with two foundational papers by the physicist Louis Michel [10] and by the mathematician Michael Field [5].

5.1 Symmetry in ODEs

Recall the example of 4 identical springs described in Chapter 4. The first idea is simple: if you consider the solution to the ODE (which is derived from Newton's laws of motion) for a given initial condition, and then reflect the initial condition one expects to get the reflected solution, and similarly for rotated initial conditions.

Now, consider V to be a representation of as group G (i.e, G is a group of matrices acting on V by matrix multiplication).

Definition 5.1. The ODE $\dot{\mathbf{x}} = f(\mathbf{x})$ for $\mathbf{x} \in V$ has ***symmetry*** G if the map f is equivariant; that is, if $f(g \cdot \mathbf{x}) = g \cdot f(\mathbf{x})$ for all $g \in G$ and $x \in V$. ✮

This definition is justified by the following property of symmetric ODEs.

Proposition 5.2. *Suppose $\dot{\mathbf{x}} = f(\mathbf{x})$ is a symmetric ODE on V and $\gamma : (-\varepsilon, \varepsilon) \to V$ is the solution with initial condition $\gamma(0) = x_0$, then the solution with initial condition $g \cdot x_0$ is $g \cdot \gamma(t)$.*

Proof: For each $t \in (-\varepsilon, \varepsilon)$, setting $\delta(t) = g \cdot \gamma(t)$ defines a smooth curve $\delta : (-\varepsilon, \varepsilon) \to V$. Clearly, $\delta(0) = g \cdot x_0$ so δ satisfies the initial condition. Furthermore, we check that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\delta(t) &= \frac{\mathrm{d}}{\mathrm{d}t} g \cdot \gamma(t) \\ &= g \cdot \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) && \text{as } g \text{ is a constant matrix} \\ &= g \cdot f(\gamma(t)) && \text{because } \gamma(t) \text{ is a solution} \\ &= f(g \cdot \gamma(t)) && \text{by equivariance} \\ &= f(\delta(t)), \end{aligned}$$

so indeed δ is a solution. ❐

Example in one variable. Consider the ODE $\dot{x} = -x$. Here, $f(x) = -x$. The general solution is $x(t) = Ae^{-t}$. If the initial condition is $x_0 = a$, then the unique (global) solution is $\gamma(t) = ae^{-t}$. This ODE has $\mathbb{Z}_2$-symmetry where $\mathbb{Z}_2 = \langle r \rangle$ and r is reflection in the origin: $r \cdot x = -x$. To see that f is $\mathbb{Z}_2$-equivariant, just check: $f(r \cdot x) = f(-x) = x$ and $r \cdot f(x) = r \cdot (-x) = x$ so $f(r \cdot x) = r \cdot f(x)$ as required. If $\gamma(t) = ae^{-t}$, then the solution with initial condition $x = r \cdot a = -a$ is $\delta(t) = (-a)e^{-t} = r \cdot \gamma(t)$ as predicted by the proposition above.

Example in two variables. We work on $\mathbb{R}^2$ with $f(x, y) = (y + x^3, x + y^3)^T$. This arises from the ODE

$$\begin{cases} \dot{x} = y + x^3 \\ \dot{y} = x + y^3 \end{cases}$$

This is impossible to solve analytically. However, let us find the symmetries of this ODE. An obvious symmetry is the one that swaps the variables x and y. This is just multiplying by the matrix $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let's check that f is equivariant with respect to this transformation. On the one hand,

$$f(\sigma(x, y)) = f(y, x) = \begin{pmatrix} x + y^3 \\ y + x^3 \end{pmatrix},$$

while on the other

$$\sigma(f(x, y)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y + x^3 \\ x + y^3 \end{pmatrix} = \begin{pmatrix} x + y^3 \\ y + x^3 \end{pmatrix},$$

showing that indeed $f \circ \sigma = \sigma \circ f$. Another symmetry is $R_\pi = -I$: a short calculation shows $f(R_\pi(x,y)) = f(-x,-y) = \begin{pmatrix} -y-x^3 \\ -x-y^3 \end{pmatrix}$ and $R_\pi f(x,y) = -\begin{pmatrix} y+x^3 \\ x+y^3 \end{pmatrix}$ are equal. So $G = \langle \sigma, R_\pi \rangle = \mathsf{D}_2' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (this is conjugate to the usual D_2). This tells us that if $\gamma(t)$ is one solution, then $\sigma\gamma(t), R_\pi\gamma(t)$ and $\sigma R_\pi \gamma(t)$ are also solutions (note that $\sigma = r_{\pi/4}$ and $\sigma R_\pi = r_{-\pi/4}$).

One of the most important properties of symmetric ODEs is given by the following theorem: it says that 'symmetry is preserved' by solutions of ODEs. Recall that $V^H = \mathrm{Fix}(H,V)$ is the subset of V of points fixed by the elements of the subgroup H.

Theorem 5.3 (Conservation of Symmetry). *Suppose the* ODE $\dot{\mathbf{x}} = f(\mathbf{x})$ *on* V *has symmetry group* G*. Let* $H \leq G$ *be a subgroup and assume* $\mathbf{x}_0 \in V^H$*. Let* $\gamma(t)$ *be the unique solution with initial condition* $\gamma(0) = \mathbf{x}_0$*. Then,* $\gamma(t) \in V^H$ *for all* t *in the domain of* γ*.*

This and Proposition 5.2 are two examples of the symmetry principle in the context of differential equations: the proposition states that if an ODE has a given symmetry then so does its set of solutions, while the theorem states that if an initial value problem has a certain symmetry, then so does the (unique) solution.

Proof: Write $D \subset \mathbb{R}$ for the domain of γ. For a fixed $h \in H$, let $\delta(t) := h \cdot \gamma(t)$ for $t \in D$. By Proposition 5.2, $\delta(t)$ is the solution with initial value $\delta(0) = h \cdot x_0 = x_0$. Since γ and δ are both solutions satisfying the same initial condition, the uniqueness theorem implies that $\delta(t) = \gamma(t)$; that is, $h \cdot \gamma(t) = \gamma(t)$ for all $t \in D$. Since $h \in H$ is arbitrary, $\gamma(t) \in V^H$ for all $t \in D$. ❐

In the example above in $\mathbb{R}^2$, this tells us that if the initial condition is fixed by σ, so is of the form (x_0, x_0), then the solution will also be fixed by σ: it will satisfy $y(t) = x(t)$ for all t for which it is defined. And indeed putting $y = x$ gives $\dot{x} = x + x^3$, which can be integrated to give $x(t) = y(t) = \pm\frac{1}{\sqrt{Ce^{-2t}-1}}$ (where the sign $\pm$ and the constant of integration C depend on the initial condition x_0).

5.2 Coupled cell systems

Coupled cell systems are systems of ODEs where each cell corresponds to 1 variable, and they interact so that the evolution (or rate of change) of each cell depends on its own state as well as the state of some of the other cells. This point of view is particularly useful when there is symmetry in the system which permutes the cells (ie, some of the cells are identical).

In applications, the variables could correspond to concentrations of chemicals in a reaction, to electronic components in a circuit, to excitation levels of biological cells in an organism, to population levels in an ecological system, ...

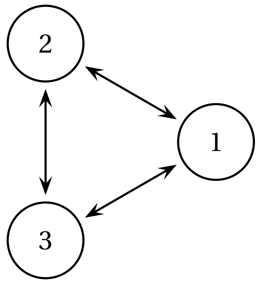
One usually assumes that if none of the cells are 'excited' (ie, all variables are equal to zero) then the system is in equilibrium.

Examples 5.4. Here we give two examples of coupled cell systems, with 3 and 4 cells respectively.

3 identical coupled cells: Consider the system of ODEs

$$\begin{cases} \dot{x}_1 &= -x_1 + x_2^2 + x_3^2 \\ \dot{x}_2 &= -x_2 + x_1^2 + x_3^2 \\ \dot{x}_3 &= -x_3 + x_2^2 + x_3^2. \end{cases}$$

The ***cell-diagram*** for this system is:

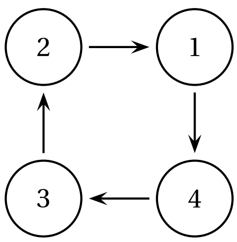


In the cell diagram, each cell has one variable: cell 1 corresponds to x_1, cell 2 to x_2, and so on. An arrow from cell A to cell B means that information flows from A to B, which in differential equations means that the evolution of B depends on the state of A (i.e $\dot{x}_B$ depends on x_A). In this example the arrows point both ways as cell 1 depends on cell 2 and vice versa. This system has S_3 symmetry, acting by permuting the cells.

4 identical coupled cells: Consider the system of 4 coupled ODEs,

$$\begin{cases} \dot{x}_1 &= -2x_1 + x_2^2 \\ \dot{x}_2 &= -2x_2 + x_3^2 \\ \dot{x}_3 &= -2x_3 + x_4^2 \\ \dot{x}_4 &= -2x_4 + x_1^2. \end{cases}$$

cell-diagram:



Notice that the arrows are unidirectional in this example. One can check (by substituting x_2 for x_1, x_3 for x_2 etc) that this system is invariant under the cyclic group $\mathbb{Z}_4 = \langle(1234)\rangle$, and is not invariant under any other permutation.

The general definition of symmetry for a coupled cell system (or network) is the following.

Definition 5.5. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and consider a family of n functions $f_i : \mathbb{R}^n \to \mathbb{R}$. The coupled cell system $\dot{x}_i = f_i(\mathbf{x})$ for $i = 1, \ldots, n$ has ***symmetry*** G if the map (or vector field)

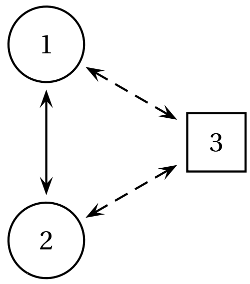
$$\begin{array}{rcl} F : \mathbb{R}^n & \to & \mathbb{R}^n \\ \mathbf{x} & \mapsto & (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) \end{array}$$

is G-equivariant. Moreover, the cells are said to be ***identical*** if G acts transitively on the set of cells. ☆

Thus in both of the examples above, the cells are identical.

Example 5.6. where the cells are *not* identical. Consider the coupled cell system,

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 + x_2^2 + x_3 \\ \dot{x}_2 & = & -x_2 + x_1^2 + x_3 \\ \dot{x}_3 & = & -x_3 + x_1 x_2. \end{array} \right.$$



Notice that exchanging x_1 and x_2 leaves the system unchanged, but any permutation with x_3 does not. The symmetry group is therefore just $G = S_2 = \mathbb{Z}_2 = \{e, (1\ 2)\}$, and the cells are not identical. In the cell-diagram we use different arrow types to show that the couplings are different, and even different boxes to show cell 3 is not identical to cells 1 and 2 (which are identical).

Recall that a stabilizer subgroup $H \le G$ is ***axial*** if $\dim(V^H) = 1$ where V is a representation of G. In these systems, the first analysis is to search for axial subgroups. For instance, consider the system in the plane,

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x + y^3 \\ \dot{y} & = & -y + x^3 \end{array} \right. \tag{5.1}$$

This system has symmetry group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (12), R_\pi \rangle$. (Geometrically, in the plane, the tranposition $(1\ 2)$ corresponds to the reflexion $r_{\pi/4}$, so the group is conjugate to D_2.) The axial subgroups are $H_1 := \langle (1\ 2) \rangle = \langle r_{\pi/4} \rangle$ and $H_2 := \langle r_{-\pi/4} \rangle$. The corresponding fixed point sets are $V^{H_1} = \{(x, x) \mid x \in \mathbb{R}\}$ and $V^{H_2} = \{(x, -x) \mid x \in \mathbb{R}\}$ respectively. So H_1 and H_2 are both axial subgroups. On V^{H_1}, the system (5.1) reduces to

$$\dot{x} = -x + x^3.$$

This can be solved using separation of variables: integrate

$$\int \frac{dx}{x^3 - x} = \int dt.$$

One obtains, $x(t) = y(t) = \frac{1}{\sqrt{Ce^{2t}+1}}$, where the constant of integration C depends on the initial condition. On V^{H_2}, (5.1) reduces to

$$\dot{x} = -x - x^3 \tag{5.2}$$

and this can be easily solved as well, with $x(t) = -y(t) = \frac{1}{\sqrt{Ce^{2t}-1}}$. Solving each of these give some special solutions of the original system (it tells us nothing about solutions outside of these fixed point spaces, but it's a start to understanding the system).

In a previous example, we looked at the following system of 4 coupled cells:

$$\begin{cases} \dot{x}_1 &=& -2x_1 + x_2^2 \\ \dot{x}_2 &=& -2x_2 + x_3^2 \\ \dot{x}_3 &=& -2x_3 + x_4^2 \\ \dot{x}_4 &=& -2x_4 + x_1^2. \end{cases} \tag{5.3}$$

It has symmetry group $G = \mathbb{Z}_4 = \langle(1234)\rangle$. The subgroups of G are

$$\mathbb{Z}_4, \quad \mathbb{Z}_2 = \langle(13)(24)\rangle \quad \text{and the trivial subgroup } \mathbb{1}.$$

Their respective fixed point sets are as follows. First, $V^{\mathbb{1}} = \mathbb{R}^4$ (then $\mathbb{1}$ is clearly not axial). Next, $V^{\mathbb{Z}_2} = \{(x_1, x_2, x_3, x_4) \mid x_1 = x_3, x_2 = x_4\}$. The latter is 2-dimensional with basis $\{(1,0,1,0)^T, (0,1,0,1)^T\}$ so $\mathbb{Z}_2$ is not axial either. The remaining subgroup is $\mathbb{Z}_4$ itself. Its fixed point set is $V^{\mathbb{Z}_4} = \mathrm{Span}\big((1,1,1,1)^T\big)$ and is thus 1-dimensional. Thus $\mathbb{Z}_4$ is axial. On $V^{\mathbb{Z}_4}$, the system (5.3) reduces to $\dot{x}_1 = -2x_1 + x_1^2$ which can be solved again by using separation of variables. In particular, there are 2 equilibrium points on this axis. Indeed, those equilibria correspond to the solutions of $0 = -2x_1 + x_1^2 = x_1(-2 + x_1)$. The axial equilibria are therefore $(0,0,0,0)^T$ and $(2,2,2,2)^T$.

5.3 Problems

5.1 Let $\mathbb{Z}_2 = \langle r\rangle$ act on $\mathbb{R}$ by $r \cdot x = -x$. Show that the differential equation $\dot{x} = \sin(2x)$ has symmetry group $\mathbb{Z}_2$. Let $x(t)$ be the solution with initial value $x(0) = 1$, and let $u(t)$ be the solution with initial value $u(0) = -1$. How are $x(t)$ and $u(t)$ related?

5.2 Let D_3 be the usual dihedral subgroup of order 6 of $\mathsf{O}(2)$, generated by r_0 and $R_{2\pi/3}$. Consider the system of ODEs

$$\begin{cases} \dot{x} &=& x + x^2 - y^2 \\ \dot{y} &=& y - 2xy \end{cases}$$

(a) Show this system has D_3 symmetry.
(b) List all three axial subgroups of D_3. By choosing one of these, find all equilibria of this system with axial symmetry, and explain briefly why it is enough to consider only one of the axial subgroups.

5.3 Let L be an $n \times n$ matrix, and consider the first order ODE $\dot{\mathbf{x}} = L\mathbf{x}$ on $\mathbb{R}^n$. Show this is equivariant for a linear action of a group G if and only if L commutes with all the matrices in the representation of G.

5.4 Consider the following family of system of ODEs in the plane

$$\begin{cases} \dot{x} &= ax + x^2 - y^2 \\ \dot{y} &= ay - 2xy \end{cases}$$

Here $a \in \mathbb{R}$ is a parameter. This is similar to the previous question, and the system has D_3 symmetry. Describe the bifurcations of equilibrium points that occur on the lines of symmetry as a is varied through $a = 0$.

5.5 Consider the similar system with symmetry D_4:

$$\begin{cases} \dot{x} &= x + x^3 - 3xy^2 \\ \dot{y} &= y - 3x^2y + y^3. \end{cases}$$

(a) Show this system has D_4 symmetry.
(b) List all axial subgroups of D_4, and find all equilibria of this system with axial symmetry. Explain briefly why in this case it is *not* enough to consider only one of the axial subgroups.

5.6 Check that the two systems in Examples 5.4 have symmetry S_3 and $\mathbb{Z}_4$ respectively.

5.7 Consider the following system of ordinary differential equations,

$$\begin{cases} \dot{x} &= -x + yz^2 \\ \dot{y} &= -y + xz^2 \\ \dot{z} &= z(1 + xy - z^2). \end{cases} \tag{$*$}$$

Consider the action of the group $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i). Show that the matrices A, B, C do indeed generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (you need to show that $A^2 = I$ etc, and A, B, C all commute).

(ii). Show that the system $(*)$ has symmetry G.

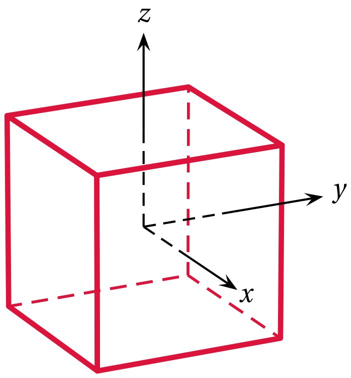
(iii). Deduce that the x-y plane and the z-axis are each invariant under the evolution of the system, stating carefully any results used.

(iv). Can you find other invariant subspaces?

(v). Find all the equilibrium points that lie on these subspaces.

(vi). Find the unique solution to this system with initial value $(x, y, z) = (1, 1, 0)$. What is the limit as $t \to \infty$ of this solution?

5.8† The octahedral group $\mathbb{O}_h$ is the group of all symmetries of the cube (including reflections). With vertices at the 8 points $(\pm1, \pm1, \pm1)$, it is generated as follows.



Generators:

$$R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_d = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$r_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Here R_z is a rotation about the z-axis by $\pi/2$, R_d is a rotation by $2\pi/3$ about the diagonal $x = y = z$, and r_z is the reflection in the x-y plane.

Consider the following family of potential functions in 3-D:

$$V = \lambda\left(x^2 + y^2 + z^2\right) - 2\left(x^4 + y^4 + z^4\right) + 3\left(x^2 y^2 + z^2 x^2 + z^2 y^2\right),$$

(this is an approximation to the system of 8 identical springs each attached to the vertex of a cube, and all attached to a common particle).

(i) Show V has symmetry $\mathbb{O}_h$ (it is enough to show it is invariant under the 3 given generators).

(ii) Show that the lines $L_1 = \{(0,0,z) \mid z \in \mathbb{R}\}$ and $L_2 = \{(x,x,0) \mid x \in \mathbb{R}\}$, and $L_3 = \{(x,x,x) \mid x \in \mathbb{R}\}$, are all 1-dimensional fixed point spaces, and find the corresponding axial subgroups. [Hint: sketch each of these lines on the figure with the cube.]

(iii) Find critical points (equilibria) occurring in these 1-dimensional fixed-point subspaces, and describe how these appear/disappear as λ varies (i.e., the bifurcations involved).

(iv) Find the other 1-dimensional fixed point spaces (all others are equivalent under the summery group $\mathbb{O}_h$ to L_1, L_2 or L_3), and list the corresponding equilibrium points.