Chapter 4

The symmetry principle

> *Lorsque certaines causes produisent certains effets, les éléments de symétrie des causes doivent se retrouver dans les effets produits.*
>
> *Pierre Curie, 1894*
>
> Where certain causes produce certain effects, the symmetries of the causes are found in those effects.

This quote (with my translation) is known as *Curie's principle*, and is named after the famous physicist Pierre Curie[1] who was interested in the effects of symmetry, There are many philosophical discussions about the truth of this principle, mostly based on interpreting the words 'cause' and 'effect'. A mathematical version which is less contentious can be stated thus:

The symmetries of a problem are found in the solution.

It is important to emphasize that this refers to 'the solution', not 'each solution'. For a simple example to illustrate what I mean, consider the problem of solving the equation $x^2 = 4$ for $x \in \mathbb{R}$. This has $\mathbb{Z}_2$ symmetry generated by $x \mapsto -x$. However the one solution $x = 2$ does not have this symmetry. On the other hand, 'the' solution to the problem is the *set* $\{2, -2\}$, which does have the same symmetry as the problem.

Thus, a less ambiguous and more modern phrasing would be,

> ***The symmetry principle:***
>
> *if a problem has symmetry then the set of solutions shares that symmetry*

This is a 'principle' and not a 'theorem' because there is no general definition of a 'problem'. However, as we will see in this and following chapters, as soon as one

[1] Pierre Curie, 1859–1906; physicist, winner of the Nobel Prize for Physics in 1903 (and husband of Marie Curie). The Nobel prize (won jointly with Marie Curie) was for the discovery of radium, but he also made important contributions to crystallography and he discovered the piezoelectric effect. This quote is taken from the Œuvres de Pierre Curie, Gauthier-Villars (1908), p.119.

specifies what a 'problem' is, one can turn it into a theorem. For example, finding solutions of polynomial equations like the example above, or solving ordinary differential equations, are both types of 'problem' where a general theorem can be stated.

Almost all of the various realizations of the symmetry principle rely on having an equivariant map: recall (Definition 1.12) that a map $\phi : V \to W$ is *equivariant* if

$$\phi(g \cdot \mathbf{v}) = g \cdot \phi(\mathbf{v}) \quad (\text{for all } \mathbf{v} \in V \text{ and all } g \in G).$$

The property of equivariant maps that underlies the symmetry principle is this, and given its importance it has a surprisingly simple proof:

Theorem 4.1. *Let $\phi : V \to W$ be equivariant. Let $\mathbf{v} \in V$ and $\mathbf{w} = \phi(\mathbf{v})$. Then, $G_{\mathbf{v}} \le G_{\mathbf{w}}$. Equivalently, if $g \cdot \mathbf{v} = \mathbf{v}$, then $g \cdot \mathbf{w} = \mathbf{w}$.*

Proof: Let $g \in G_{\mathbf{v}}$. Then $g \cdot \mathbf{w} = g \cdot \phi(\mathbf{v}) = \phi(g \cdot \mathbf{v}) = \phi(\mathbf{v}) = \mathbf{w}$ so $g \in G_{\mathbf{w}}$. ❒

4.1 Invariant functions

For this section, we will restrict attention to actions of groups on vector spaces, because we want to do calculus.

Definition 4.2. Let V be a finite dimensional vector space. A ***linear action*** of a group G on V is a homomorphism $\rho : G \to \mathsf{GL}(V)$ where $\mathsf{GL}(V)$ is the group of invertible linear maps $V \to V$. ✯

Linear actions are also called[2] ***representations***. If we choose a basis for V, then each $\rho(g)$ is an invertible $n \times n$ matrix (where $n = \dim V$). So here, an element $g \in G$ acts on V by standard matrix multiplication; that is, $g \cdot \mathbf{v} = \rho(g)\mathbf{v}$.

In fact we will assume that the representation of G on V is an *orthogonal representation* meaning that $\rho : G \to \mathsf{O}(V)$ (that is, for each $g \in G$, $\rho(g)$ is an orthogonal matrix, so satisfies $\rho(g)\rho(g)^T = I$). (This is not really restrictive: for a finite group one can always choose a basis for which this is true.)

Examples We have already seen C_n and D_n acting on $\mathbb{R}^2$ in this way. Other examples are the symmetries of the tetrahedron and the cube in $\mathbb{R}^3$.

Definition 4.3. A function $f : V \to \mathbb{R}$ is said to be ***invariant*** if $f(g \cdot \mathbf{v}) = f(\mathbf{v})$ for every $g \in G$ and every $\mathbf{v} \in V$. ✯

[2] Representation theory is the study of representations, often over different fields, and is a substantial and active branch of algebra

Note that to show a given function is invariant under a group, it suffices to check it for the generators of the group: see the example below.

Recall that, if $f : X \to Y$ is a map, then for $y \in Y$, one denotes the corresponding ***level set*** of f by,

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

Proposition 4.4. *If $f : V \to \mathbb{R}$ is invariant, then for any $a \in \mathbb{R}$, the set $f^{-1}(a)$ is an invariant subset of V; that is, if $\mathbf{u} \in f^{-1}(a)$ then $g \cdot \mathbf{u} \in f^{-1}(a)$ for every $g \in G$.*

This is an example of the symmetry principle: the 'problem' $f(\mathbf{u}) = a$ has symmetry G (since f is G-invariant), and the proposition says that the set of solutions also has symmetry G.

Proof: Suppose $f(\mathbf{u}) = a$. We need to show that $f(g\cdot\mathbf{u}) = a$. But $f(g\cdot\mathbf{u}) = f(\mathbf{u}) = a$ since f is invariant. ❐

Example Consider the usual action of D_3 on $\mathbb{R}^2$, as discussed in Chapters 1 and 2. D_3 is generated by $R_{2\pi/3}$ and r_0. Let $f(x, y) = x^3 - 3xy^2 + x^2 + y^2$. We claim that f is invariant. To see this, it suffices to show it is unchanged under composing with r_0 and $R_{2\pi/3}$. For r_0 we get

$$\begin{aligned} f\circ r_0(x,y) &= f(x,-y) \\ &= x^3 - 3x(-y)^2 + x^2 + (-y)^2 \\ &= f(x,y). \end{aligned}$$

For $R_{2\pi/3}$, recall that $R_{2\pi/3} = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$, with $\cos(2\pi/3) = -1/2$ and $\sin(2\pi/3) = \sqrt{3}/2$. Thus,

$$\begin{aligned} f\circ R_{2\pi/3}(x,y) &= \tfrac{1}{8}(-x-\sqrt{3}y)^3 - \tfrac{3}{8}(-x-\sqrt{3}y)(\sqrt{3}x-y)^2 + \\ &\quad + \tfrac{1}{4}(-x-\sqrt{3}y)^2 + \tfrac{1}{4}(\sqrt{3}x-y)^2 \\ &= f(x,y). \end{aligned}$$

Since f is invariant under both r_0 and $R_{2\pi/3}$ it is invariant under the group they generate, which is D_3. For example, $f\circ(r_0R_{2\pi/3}) = (f\circ r_0)\circ R_{2\pi/3} = f\circ R_{2\pi/3} = f$, and $f\circ R_{4\pi/3} = (f\circ R_{2\pi/3})\circ R_{2\pi/3} = f\circ R_{2\pi/3} = f$. In Fig. 4.1 we see that the level sets (or contours) of f are invariant sets.

4.2 Critical points of invariant functions

There are many problems in applied mathematics (or physics or chemistry) where the solution is given by the critical points of a function. If that function is invariant

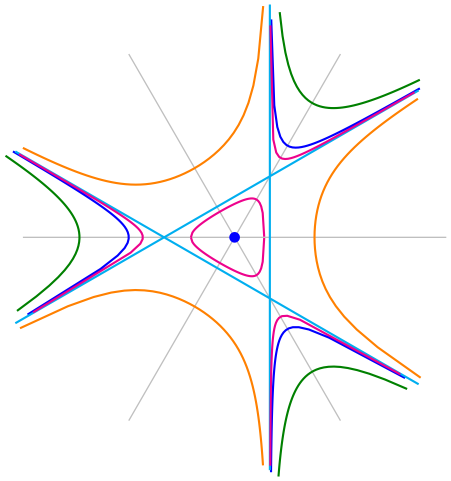

FIGURE 4.1: Level sets of the invariant function $f(x,y) = x^3 - 3xy^2 + x^2 + y^2$. The grey lines are the lines of reflection for the D_3 action. Notice that each of the coloured sets is invariant under the action of D_3. The value $4/27$ is the value of f at each of its three saddle points.

under a group action, then the symmetry principle says that the set of solutions (set of critical points) should also be invariant. We state this below in Theorem 4.5, whose proof is given on p.4.6.

Recall that a ***critical point*** of a differentiable function f is a point x for which $\frac{\partial f}{\partial x_1}(x) = \frac{\partial f}{\partial x_2}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0$.

The symmetry principle in this setting states:

Theorem 4.5. *If $f : V \to \mathbb{R}$ is invariant, then its set $C(f)$ of critical points is an invariant subset.*

The proof can be divided into a few useful intermediate results.

Lemma 4.6. *Let $f : V \to \mathbb{R}$ be a G-invariant differentiable function. Then, its gradient $\nabla f : V \to V$ is equivariant.*

Proof: We use the chain rule (see box on p. 4.5), with $h(\mathbf{v}) = g \cdot \mathbf{v} = \rho(g)\mathbf{v}$. Now the Jacobian matrix of h is just $J(h) = \rho(g)$. Thus, by (4.1),

$$\nabla(f(h(\mathbf{v}))) = \rho(g)^T \nabla f(\mathbf{v}).$$

Now we use the invariance of f, that is $f(\rho(g)\mathbf{v}) = f(\mathbf{v})$. Then, differentiating this,

$$\rho(g)^T \nabla f(\rho(g)\mathbf{v}) = \nabla f(\mathbf{v}).$$

Multiplying both sides by $\rho(g)$ gives

$$\nabla f(\rho(g)\mathbf{v}) = \rho(g)\nabla f(\mathbf{v}),$$

as required. ❐

Chain rule in several variables Let U be an open subset of $\mathbb{R}^n$, and write $\mathbf{x} = (x_1, \ldots, x_n)$ for a general point of U. The x_j are called the coordinates of the point $\mathbf{x}$.

Now suppose $f : U \to \mathbb{R}$ is a differentiable function. One writes $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ and defines its ***gradient*** as being the column vector

$$\nabla f = \operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)^T.$$

Now let $h : U \to U$ be a differentiable map which reads $h(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_n(\mathbf{x}))$ again with respect to the coordinates $(x_1, \ldots, x_n)$. The *chain rule* states that, for each $i = 1, \ldots, n$,

$$\frac{\partial}{\partial x_i}(f(h(\mathbf{x}))) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(h(\mathbf{x}))\frac{\partial h_j}{\partial x_i}(\mathbf{x}).$$

The gradient of the composed map $f \circ h : V \to \mathbb{R}$ thus reads

$$\nabla(f \circ h)(\mathbf{x}) = \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(h(\mathbf{x}))\frac{\partial h_j}{\partial x_1}(\mathbf{x}), \ldots, \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(h(\mathbf{x}))\frac{\partial h_j}{\partial x_n}(\mathbf{x})\right)^T.$$

This can be written in matrix form as,

$$\nabla(f \circ h)(\mathbf{x}) = J(h)^T \nabla f(\mathbf{x}), \tag{4.1}$$

where $J(h)$ is the Jacobian matrix of h: that is, $J(h)_{kj} = \frac{\partial h_k}{\partial x_j}$.

Example Consider the function $f(x, y) = x^3 - 3xy^2$. Then

$$\nabla f = \begin{pmatrix} 3x^2 - 3y^2 \\ -6xy \end{pmatrix}.$$

Now let $h(u, v) = (uv, 2u)$, for which $J(h) = \begin{pmatrix} v & u \\ 2 & 0 \end{pmatrix}$. Then $f \circ h(u, v) = u^3v^3 - 12u^3v$ and hence

$$\nabla(f \circ h)(u, v) = \begin{pmatrix} 3u^2v^3 - 36u^2v \\ 3u^3v^2 - 12u^3 \end{pmatrix} = \begin{pmatrix} v & 2 \\ u & 0 \end{pmatrix}\begin{pmatrix} 3x^2 - 3y^2 \\ -6xy \end{pmatrix}$$

after substituting $x = uv$ and $y = 2u$.

Proposition 4.7. *If $\phi : V \to W$ is equivariant, then $\phi^{-1}(0)$ is an invariant subset of V.*

Proof: Suppose $\mathbf{v} \in \phi^{-1}(0)$. We want to show that for any $g \in G$, $g \cdot \mathbf{v} \in \phi^{-1}(0)$. Now, $\phi(g \cdot \mathbf{v}) = g \cdot \phi(\mathbf{v}) = g \cdot 0 = 0$, so we are done. ❐

We can now prove the theorem.

Proof of Theorem 4.5: A critical point $\mathbf{v} \in C(f)$ is such that $\nabla f(\mathbf{v}) = 0$. Therefore, $C(f) = (\nabla f)^{-1}(0)$. But ∇f is equivariant (Lemma 4.6) so the result follows from Proposition 4.7. ❐

It is perhaps worth emphasizing that the theorem does *not* say that all critical points have non-trivial symmetry: this is a common misunderstanding.

Example 4.8. As in the previous example, we consider the function

$$f(x, y) = x^3 - 3xy^2 + x^2 + y^2.$$

Its gradient is given by $\nabla f(x, y) = (3x^2 - 3y^2 + 2x, -6xy + 2y)^T$. Then, $\nabla f = 0$ implies that

$$\begin{cases} 2y(1-3x) = & 0 \\ 3x^2 - 3y^2 + 2x = & 0 \end{cases} \tag{4.2}$$

The set of solutions is

$$C(f) = \left\{(0,0), (-2/3, 0), (1/3, 1/\sqrt{3}), (1/3, -1/\sqrt{3})\right\}.$$

These form two orbits; namely, $\{(0,0)\}$ which has stabilizer D_3, and

$$\left\{(-2/3, 0), (1/3, 1/\sqrt{3}), (1/3, -1/\sqrt{3})\right\}.$$

In the latter set, the stabilizer of $(-2/3, 0)$ is $\mathsf{D}_1 = \{I, r_0\}$, and hence the orbit type is (D_1), and this the Burnside type of the set of critical points is

$$\mathcal{B}(C(f)) = 1(\mathsf{D}_3) + 1(\mathsf{D}_1).$$

4.3 Fixed point subspaces

Let $\rho : G \to \mathsf{GL}(V)$ be a representation of G on a (finite dimensional) vector space V, and let $H \le G$.

Definition 4.9. The ***fixed point subspace*** of H is

$$V^H = \mathrm{Fix}(H, V) = \{\mathbf{v} \in V \mid h \cdot \mathbf{v} = \mathbf{v} \text{ for any } h \in H\}.$$

This is the same as $\{\mathbf{v} \in V \mid H \le G_{\mathbf{v}}\}$. ✯

Example. Consider the linear representation of D_3 on $\mathbb{R}^2$. Let $H = \langle r_0 \rangle$. Then, $\mathrm{Fix}(H, \mathbb{R}^2)$ is the $x-$axis. Notice that this is a vector subspace of $\mathbb{R}^2$, and that is a general fact:

Proposition 4.10. *V^H is a vector subspace of V.*

Proof: First, note that $0 \in V^H$. Suppose $\mathbf{u}, \mathbf{v} \in V^H$. We want to show $\lambda\mathbf{u} + \mu\mathbf{v} \in V^H$ for all $\lambda, \mu \in \mathbb{R}$. Let $h \in H$ and consider

$$\begin{aligned} h \cdot (\lambda\mathbf{u} + \mu\mathbf{v}) &= \rho(h)(\lambda\mathbf{u} + \mu\mathbf{v}) \\ &= \lambda\rho(h)\mathbf{u} + \mu\rho(h)\mathbf{v} \text{ by linearity} \\ &= \lambda\mathbf{u} + \mu\mathbf{v}. \end{aligned}$$

But this holds for arbitrary $h \in H$ and therefore $\lambda\mathbf{u} + \mu\mathbf{v} \in V^H$. ❒

The following theorem, due to Richard Palais, is crucial to the use of symmetry in the remainder of this chapter.

Theorem 4.11 (Principle of Symmetric Criticality). *Let $f : V \to \mathbb{R}$ be an invariant differentiable function where V is a representation of G. Let $H \le G$ and suppose $\mathbf{v} \in V^H$. Then, $\mathbf{v}$ is a critical point of f if and only if it is a critical point of the restriction $f|_{V^H} : V^H \to \mathbb{R}$.*

Proof: Since f is an invariant function, its gradient ∇f is equivariant (Lemma 4.6). If $\mathbf{v} \in V^H$, then applying Theorem 4.1 to $\mathbf{w} = \nabla f(\mathbf{v})$ shows that $\nabla f(\mathbf{v}) \in V^H$ (see Fig. 4.2). There remains to interpret this in terms of the statement of the theorem.

Let $k = \dim(V^H)$ and choose a basis $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$ for V with $\mathsf{e}_1, \ldots, \mathsf{e}_k \in V^H$ and $\mathsf{e}_{k+1}, \ldots, \mathsf{e}_n \in (V^H)^\perp$. Then

$$\nabla f(\mathbf{v}) = (a_1, \ldots, a_k, 0, \ldots, 0)^T \in V^H$$

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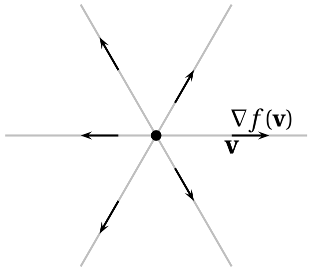

FIGURE 4.2: At points $\mathbf{v}$ of V^H, $\nabla f(\mathbf{v})$ also belongs to V^H

for some real numbers $a_1,\ldots,a_k$. So $\mathbf{v}$ is a critical point of f if and only if

$$\frac{\partial f}{\partial x_1}(\mathbf{v}) = \cdots = \frac{\partial f}{\partial x_k}(\mathbf{v}) = 0. \tag{4.3}$$

On the other hand $f|_{V^H} = f(x_1,\ldots,x_k,0,\ldots,0)$ which has a critical point at $\mathbf{v}$ if and only if (4.3) holds. ❐

Example 4.12. Let $G = \mathsf{D}_3$ and $H = \mathsf{D}_1 = \langle r_0 \rangle$, and consider $f(x,y) = x^3 - 3xy^2 + x^2 + y^2$, which is an invariant function (see Fig. 4.1). Its gradient is

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 - 3y^2 + 2x \\ 2y(1-3x) \end{pmatrix}.$$

Now, V^H is the line with equation $y = 0$ and substituting $y = 0$ into ∇f gives

$$\nabla f(x,0) = \begin{pmatrix} 3x^2 + 2x \\ 0 \end{pmatrix}.$$

That is $\partial f/\partial y = 0$ so a point of the form $(x,0)$ is a critical point if and only if $\frac{\partial f}{\partial x} = 0$ at that point, i.e. it is a critical point of $f|_{V^H}$, as predicted by the Principle of Symmetric Criticality above. The same occurs for the other fixed point spaces, see Fig. 4.2 and Problem 4.6.

In the following example, we consider a simple mechanical example consisting of a mass connected to 4 springs, all with the symmetry of a square. The detailed calculations are not simple but also not so important: it is the principles involved that are important.

Example 4.13. Consider a system with four identical springs $S_1,\ldots,S_4$ in the plane, each attached to a vertex of a square and the other ends all attached to a single particle, which is free to move in the plane; see Fig. 4.3. Let (x,y) be the coordinates of the particle. We assume all the springs are identical, in which case this system

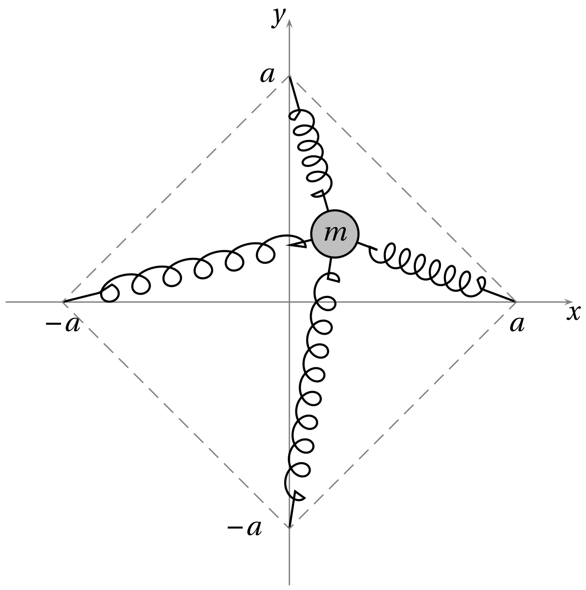

FIGURE 4.3: Arrangement of 4 springs with D_4 symmetry (see Example 4.13)

has D_4 symmetry. We want to find the equilibrium configurations of this system. Clearly, $(x, y) = (0,0)$ is one equilibrium point (by symmetry!); are there any others?

A spring has a natural length ℓ. If the spring is displaced by a length x, the potential energy is $V = \frac{1}{2}kx^2$ because Hooke's law says that $\mathbf{F} = -k\mathbf{x}$ and $\mathbf{F} = -\frac{\partial V}{\partial x}$. Take $k = \ell = 1$ (corresponding to choices of physical units of length and mass). For each spring S_i, let V_i be the potential energy associated to it. We find

$$\begin{aligned} V_1 &= \tfrac{1}{2}k\left(\sqrt{(x-a)^2+y^2}-1\right)^2, & V_2 &= \tfrac{1}{2}k\left(\sqrt{x^2+(y-a)^2}-1\right)^2, \\ V_3 &= \tfrac{1}{2}k\left(\sqrt{(x+a)^2+y^2}-1\right)^2, & V_4 &= \tfrac{1}{2}k\left(\sqrt{x^2+(y+a)^2}-1\right)^2. \end{aligned}$$

The total potential energy is $V = \sum_{i=1}^4 V_i$. Equilibrium points occur wherever $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$, that is, at critical points of V. Since the system is symmetric, the potential energy V is invariant. Here, $W = \mathbb{R}^2$ with its action of D_4. Note that $\mathrm{Fix}(D_4, \mathbb{R}^2) = \{(0,0)\}$. Since this is an isolated point, all partial derivatives of V vanish at $(0,0)$ by the Principle of Symmetric Criticality (PSC, Theorem 4.11). Therefore (as we know already), $(0,0)$ is an equilibrium point.

To find further critical points, we apply further the PSC. Let $H = D_1 = \langle r_0 \rangle = \{I, r_0\}$. Then $\mathrm{Fix}(H, \mathbb{R}^2) = \{y = 0\}$. The restriction of V to $\mathrm{Fix}(H, \mathbb{R}^2)$ is (letting $k = 1$),

$$V(x,0) = \frac{1}{2}(|x-a|-1)^2 + \frac{1}{2}(|x+a|-1)^2 + \left(\sqrt{x^2+a^2}-1\right)^2$$

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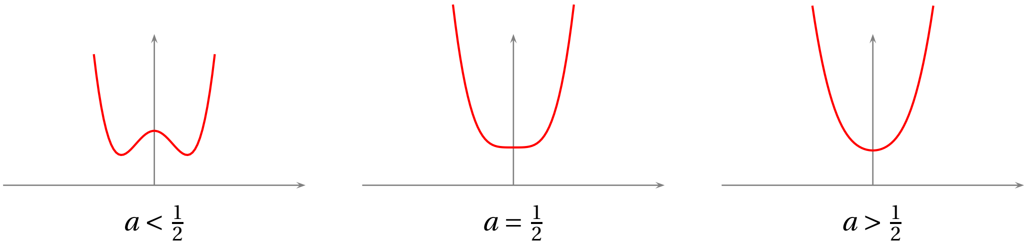

FIGURE 4.4: Graphs of the potential $V(x,0)$ for the 4-spring problem, near the equilibrium point at the origin, for different values of a

The Taylor series near $x = 0$ is

$$V(x,0) = 2(a-1)^2 + \frac{1}{a}(2a-1)x^2 + \frac{1}{4a^3}x^4 + \dots.$$

Note that if $a > \frac{1}{2}$, this has a local minimum at $x = 0$. However, if $a < \frac{1}{2}$, this has a local maximum at $x = 0$. The graphs of $V(x,0)$ with a close to 0.5 and x close to 0 are shown in Figure 4.4.

Notice how the number of critical points on the x-axis depends on the value of a (the point where the springs are attached—or equivalently it depends on the strength of the spring).

Note: the two variable Taylor series at the origin for this spring problem is

$$V(x,y) = 2(a-1)^2 + (2-\tfrac{1}{a})(x^2+y^2) + \frac{1}{4a^3}(x^4+y^4) - \frac{2}{a^3}x^2y^2 + O(6), \tag{4.4}$$

and you can see that the quadratic term vanishes if $a = \frac{1}{2}$.

In the spring problem above, the equilibrium at $(x, y) = (0,0)$ has full D_4 symmetry. When $a > \frac{1}{2}$ this is the only critical point (assuming the springs all have positive length). When $a < \frac{1}{2}$, there are two other equilibria on the x-axis (and by rotational symmetry also on the y-axis), and these 'new' equilibria have less symmetry. Indeed, they have symmetry $\mathsf{D}_1 = \langle r_0 \rangle$. This is an example of the phenomenon of ***spontaneous symmetry breaking***: a solution of an equation has symmetry G, but as some parameter is varied (here a) solutions appear nearby whose symmetry is equal to a proper subgroup of G.

(Another way to 'break' the symmetry of this system would be to move one of the points where one of the springs is attached, so that the 4 points no longer form a square, and this perturbed system would no longer have D_4 symmetry. This process is called ***forced symmetry breaking***.)

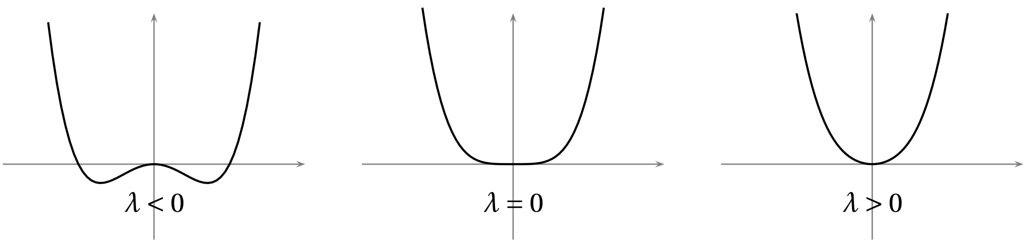

FIGURE 4.5: Graph of $\lambda x^2 + x^4$ for different values of λ

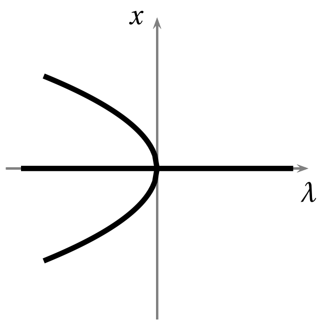

FIGURE 4.6: Bifurcation diagram showing the critical points for the family $f_\lambda(x) = \lambda x^2 + x^4$

4.4 Bifurcations: New critical points from old

The function $f_\lambda(x) = \lambda x^2 + x^4$ has derivative $f'_\lambda(x) = \frac{\partial f_\lambda}{\partial x} = 2x(\lambda + 2x^2)$. Its critical points are thus given by the set of equations

$$\begin{cases} x = 0 \\ \lambda + 2x^2 = 0. \end{cases}$$

This type of bifurcation, with a transition from 3 to 1 or 1 to 3 critical points is called a ***pitchfork bifurcation***. The so-called 'bifurcation diagram' is shown in Fig. 4.6.

We begin with problems with one variable and one parameter: Consider critical points of $f_\lambda(x)$ where $\lambda \in \mathbb{R}$ is a parameter and $x \in \mathbb{R}$ is a variable. How does the set of critical points changes as λ changes? An example—the pitchfork bifurcation—is discussed above.

For our purposes, we may assume $f'_\lambda(0) = 0$ for any $\lambda \in \mathbb{R}$ i.e, $x = 0$ is always a critical point. This implies the Taylor series of f_λ about 0 is of the form

$$f_\lambda(x) = f_\lambda(0) + \tfrac{1}{2}a(\lambda)x^2 + \tfrac{1}{3!}b(\lambda)x^3 + \cdots, \tag{4.5}$$

for some smooth functions $a(\lambda)$, $b(\lambda)$, ... etc. Critical points arise whenever the deriva-

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tive with respect to x vanishes:

$$f'_\lambda(x) = a(\lambda)x + \frac{1}{2}b(\lambda)x^2 + \cdots = 0.$$

Clearly $x = 0$ is always a solution to this equation (as we required). If $a(\lambda_0) \neq 0$, then it is the only solution in a (perhaps small) neighbourhood of 0 and that remains the case even as λ is varied near λ_0. On the other hand, if $a(\lambda_0) = 0$ one expects some change, or bifurcation to occur as λ is varied. To ensure this, let us assume that $a'(\lambda_0) = \frac{d}{d\lambda}a(\lambda_0) \neq 0$.

Theorem 4.14. *Let $f_\lambda(x)$ be a smooth 1-parameter family of functions and assume that $f'_\lambda(0) = 0$ for each λ (as above), and write f in the form (4.5). Suppose moreover that $a(\lambda_0) = 0$ but $a'(\lambda_0) \neq 0$. Then $x = 0$ is a critical point for each λ and there is another curve of critical points of f_λ in the x-λ plane, passing through $(x, \lambda) = (0, \lambda_0)$.*

The proof (which we don't give) is an application of the Implicit Function Theorem. Note that $a(\lambda) = f''_\lambda(0)$, and so $a'(\lambda) = \frac{\partial^3}{\partial\lambda\partial x^2} f_\lambda(0)$.

Observation: If the representation V is such that $V^G = \{0\}$, then 0 is a critical point of every invariant function (see Problem 4.5). As a general consequence, if f_λ is a 1-parameter family of invariant functions and $H \le G$ is such that $\dim(V^H) = 1$ (for instance, $V = \mathbb{R}^2$, $G = \mathsf{D}_4$ and $H = \langle r_0 \rangle$ then V^H is the x-axis), then we can use Theorem 4.14 together with the PSC to find critical points in V^H, i.e., critical points with symmetry H.

Definition 4.15. Let V be a representation of G, and suppose H is such that $\dim V^H = 1$, then H is said to be an ***axial subgroup***. ✩

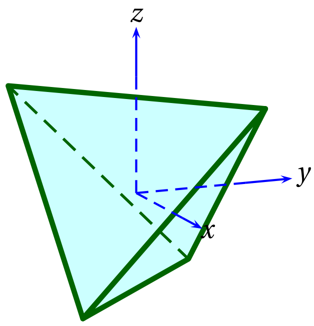
Thus these 1-variable methods apply to axial subgroups. If G is D_n acting on the plane, then the axial subgroups are all subgroups generated by a reflection, such as $H = \langle r_0 \rangle$; the corresponding fixed point space V^H being the line of reflection.

Note: Let $g \in G$, which is acting on $V = \mathbb{R}^n$ by linear (orthogonal) transformations, or matrices. The fixed point subspace of g is the set of vectors $\mathbf{x}$ satisfying $g \cdot \mathbf{x} = \mathbf{x}$. This is the same as the set of eigenvectors with eigenvalue 1.

4.5 A 3-dimensional example: the Tetrahedron

A regular tetrahedron is a solid with 4 vertices, 4 faces and 6 edges, and each of the 4 faces is an equilateral triangle. See the figure below.

It is useful for calculations to use a particular orientation of the tetrahedron, with the x, y and z axes cutting through the mid-points of 3 of the edges, shown on the right. The 4 vertices are



$$\begin{aligned} V_1 &= (1,1,1), & V_2 &= (-1,-1,1), \\ V_3 &= (1,-1,-1), & V_4 &= (-1,1,-1). \end{aligned}$$

Note that these are the points of the form $(\pm1,\pm1,\pm1)$ with an even number of -1s.

Indeed, the 8 points of the form $(\pm1,\pm1,\pm1)$ form the vertices of a cube, and this tetrahedron sits nicely in that cube. Moreover, the other 4 points, those with an odd number of -1s, make up the vertices of another ('dual') tetrahedron inscribed in the cube. See Figure 4.7. Notice (for later) that a rotation by $\pi/2$ about any of the axes exchanges the two tetrahedra.

The symmetry group Let $V = \{V_1, V_2, V_3, V_4\}$ be the set of 4 vertices listed above. The full symmetry group of the tetrahedron is traditionally denoted $\mathbb{T}_d$ (the symbol $\mathbb{T}$ denotes just the rotational symmetries). Since any symmetry of the tetrahedron necessarily permutes the vertices, $\mathbb{T}_d$ acts on the set V, giving a homomorphism

$$\rho_V : \mathbb{T}_d \longrightarrow S_4.$$

I claim that this is an isomorphism. In the first place it is clearly injective, for the only symmetry that fixes all 4 vertices is the identity. Why is it surjective? To see this, recall that S_4 is generated by transpositions, so if we show that each transposition is in the image of ρ_V then we are done.

The transposition $(1\ 2)$ (swapping V_1 and V_2) is obtained by a reflection in the

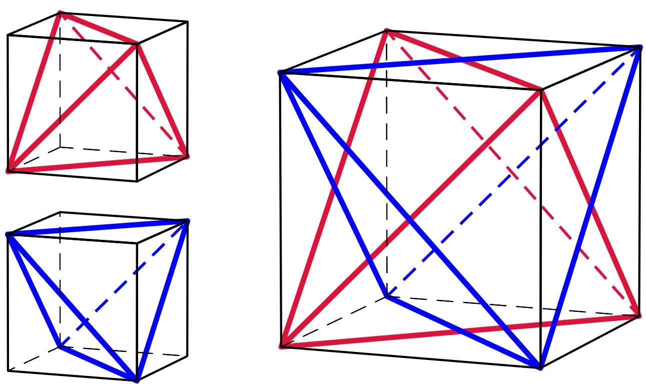


FIGURE 4.7: Two 'dual' tetrahedra inscribed in a cube

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plane $x+y=0$. As a matrix this is

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{T}_d.$$

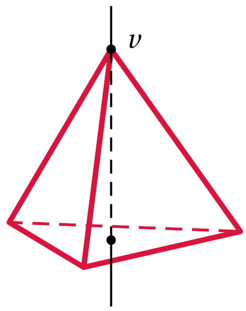
The other transpositions can be found as other reflections.

Thus the symmetry group of the tetrahedron $\mathbb{T}_d$ is isomorphic to S_4, and consequently is of order 24 (this can also be proved using the orbit-stabilizer theorem).

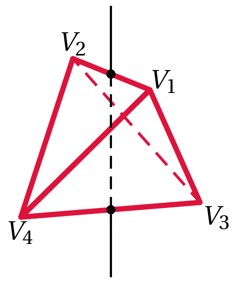
Axial subgroups To apply the methods of the previous sections, we are interested foremost in the axial subgroups, and we see there are just 2 'types' (2 conjugacy classes). We will use the isomorphism derived above to refer to elements of $\mathbb{T}_d$ by permutations of the vertices.

(1). For the first type, choose any one of the vertices, say v, and let $(\mathbb{T}_d)_v$ be its stabilizer. By the orbit-stabilizer theorem this has order $\frac{|\mathbb{T}_d|}{|\mathbb{T}_d \cdot v|} = \frac{24}{4} = 6$. Indeed, it can be identified with the subgroup S_3 obtained by leaving v fixed and permuting the other 3 vertices. As a geometric subgroup of $\mathbb{T}_d$ it is isomorphic to the group D_3 of symmetries of the equilateral triangle, and in particular of the face opposite to v.

Since the different vertices are all in the same orbit, it follows that the stabilizers of two different vertices are conjugate subgroups of $\mathbb{T}_d$ (or of S_4). (See Proposition 1.6.)



(2). The other type of axial subgroup is obtained using as axis the line through the mid-points of opposite edges (there are 3 such lines, shown as the x- y- and z-axes in one of the earlier figures). Let V_1, V_2 be the vertices on one edge and V_3, V_4 the vertices on the opposite edge. Rotating by π about this axis acts as $(12)(34)$, the permutation which permutes both the vertices V_1 and V_2 and the vertices V_3 and V_4. Also, (12) is the reflection in the plane containing V_3, V_4 and this axis. Similarly, (34) is the reflection in the plane containing V_1, V_2 and the axis of rotation. This group is the stabilizer of m_{12} (respectively m_{34}), the mid-points of the edge joining V_1 and V_2 (V_3 and V_4 respectively). The stabilizer $(\mathbb{T}_d)_{m_{12}} = \langle(12),(34)\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein's 4-group). There are 3 conjugate copies of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in $\mathbb{T}_d$, the others being $\langle(13),(24)\rangle$ (which fixes m_{13} and m_{24}) and $\langle(14),(23)\rangle$ (which fixes m_{14} and m_{23}).



Spring example. Consider 4 identical springs each attached to a vertex of a regular tetrahedron and all attached to a single particle. The springs have natural length ℓ

and strength $k = 1$. The coordinates of the vertices are $(1,1,1)$, $(-1,-1,1)$, $(1,-1,-1)$ and $(-1,1,-1)$. We know that the origin is an equilibrium (by the PSC: it is the only point fixed by the group $\mathbb{T}_d$). If the particle has coordinates (x,y,z), the total potential energy is given by adding together the contribution from the 4 springs:

$$\begin{aligned} V(x,y,z) &= \tfrac{1}{2}\left(\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}-\ell\right)^2 \\ &\quad +\tfrac{1}{2}\left(\sqrt{(x+1)^2+(y+1)^2+(z-1)^2}-\ell\right)^2 \\ &\quad +\tfrac{1}{2}\left(\sqrt{(x+1)^2+(y-1)^2+(z+1)^2}-\ell\right)^2 \\ &\quad +\tfrac{1}{2}\left(\sqrt{(x-1)^2+(y+1)^2+(z+1)^2}-\ell\right)^2. \end{aligned} \tag{4.6}$$

Equilibrium points occur when $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0$. However, this is too nasty to solve by brute-force calculation. Instead, we use the symmetry! As a first observation, the fixed point set $V^{\mathbb{T}} = \{0\}$ (for $V = \mathbb{R}^3$). Therefore, $(0,0,0)$ is a critical point of the potential V.

Next, we use the axial subgroups. For the vertex $v = (1,1,1)$, let $H = (\mathbb{T}_d)_v$ be its stabilizer. The fixed point set V^H is the line through v and the origin. This line has equation $x = y = z$. We substitute in (4.6) to get

$$V(x,x,x) = \frac{1}{2}\left(\sqrt{3}|x-1|-\ell\right)^2 + \left(\sqrt{2(x+1)^2+(x-1)^2}-\ell\right)^2 + \frac{1}{2}\left(\sqrt{3}|x+1|-\ell\right)^2 \tag{4.7}$$

Because this line is the fixed point subspace, the PSC tells us that to find critical points of V that lie on this line we only need to solve $\frac{d}{dx}V(x,x,x) = 0$, we don't need to care about the derivatives in the other directions. The bifurcation diagram for critical points on the axis $x = y = z$ is shown on the left hand side of Fig. 4.8 (it is computed numerically, which explains why the point where the curve crosses the ℓ-axis is not well-resolved). The bifurcation where the two curves cross is an example of a ***transcritical bifurcation***.

For the second axial subgroup consider the subgroup $\langle (12), (34)\rangle$. The fixed point set is the z-axis so $\{x = y = 0\}$. Substituting this in (4.6), yields

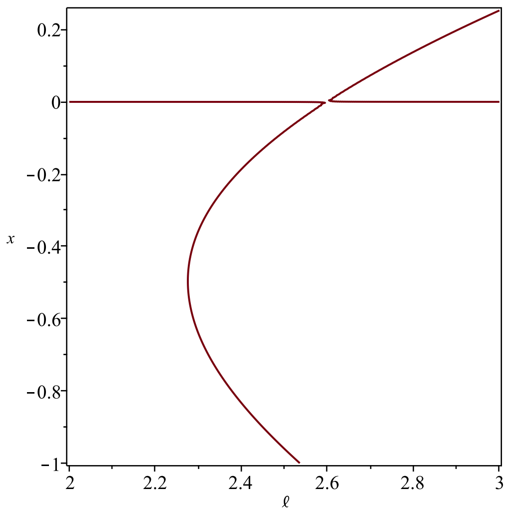
$$V(0,0,z) = (|z-1|-\ell)^2 + (|z+1|-\ell)^2 \tag{4.8}$$

and by the PSC we only need to find where $\frac{d}{dz}V(0,0,z) = 0$. The bifurcation diagram (found numerically by computer) is shown on the right in Fig. 4.8. The point where the two curves cross is a pitchfork bifurcation, because $V(0,0,z) = V(0,0,-z)$ in (4.8).

To study the two bifurcations in more detail consider the Taylor series of V at the origin (omitting the constants):

$$V = \lambda(x^2+y^2+z^2) + \frac{4\sqrt{3}\ell}{9}xyz - \frac{2\sqrt{3}\ell}{81}(x^4+y^4+z^4) + \frac{4\sqrt{3}\ell}{27}(x^2y^2+y^2z^2+z^2x^2) + O(5)$$

where $\lambda = 2 - \frac{4\sqrt{3}\ell}{9}$. Note that V has a local minimum at $(0,0,0)$ if $\lambda > 0$ i.e $\ell < \frac{3\sqrt{3}}{2}$ and a local maximum if $\lambda < 0$ i.e if $\ell > \frac{3\sqrt{3}}{2}$. So, the origin is a stable equilibrium if $\ell < \frac{3\sqrt{3}}{2}$ and an unstable equilibrium if the inequality is reversed.



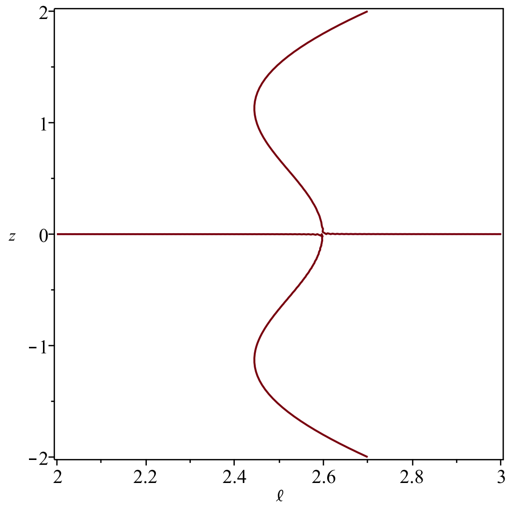


FIGURE 4.8: Bifurcation diagrams for critical points of $V(x,x,x)$ and $V(0,0,z)$ respectively; in both cases the bifurcation occurs when $\ell = 3\sqrt{3}/2 \approx 2.6$

On the diagonal,

$$V(x,x,x) = 3\lambda x^2 + \frac{4\sqrt{3}\ell}{9}x^3 + \frac{10\sqrt{3}\ell}{27}x^4 + O(5)$$

On the axis

$$V(0,0,z) = \lambda z^2 - \frac{2\sqrt{3}\ell}{81}z^4 + O(6).$$

Notice how the second is even in z while the first is not even in x (because of the x^3 term). That accounts for the symmetry in the diagram on the right that is absent in the one on the left. There is a geometric/group theoretic explanation for this difference which we will see in the next chapter (it's related to the normalizer of the stabilizer subgroup).

4.6 Problems

4.1 Let $\mathbb{Z}_3 = \{0,1,2\}$ with addition modulo 3, and let $\omega = e^{2\pi i/3}$ (note that $\omega^3 = 1$). Consider the action of $\mathbb{Z}_3$ on the complex plane $\mathbb{C}$ defined by

$$n \cdot z = \omega^n z$$

(i) Show first this is indeed an action. (ii) Show that the equation $z^3 = 8$ has symmetry $\mathbb{Z}_3$ and that the set of solutions also has this symmetry.

4.2 Let the group G act on two sets X and Y, and suppose that $\phi : X \to Y$ is equivariant. If in addition we suppose ϕ is a bijection, show that ϕ^{-1} is also equivariant.

4.3 Let V be a representation of G, and let $A : V \to V$ be a linear map (a matrix), which is equivariant. Recall that if $\mathbf{v} \neq \mathbf{0}$ satisfies $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$ one says $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.

(i). Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then so is $g \cdot \mathbf{v}$ for each $g \in G$.

(ii). Let E_λ be the λ-eigenspace of A,

$$E_\lambda = \{\mathbf{v} \in V \mid A\mathbf{v} = \lambda\mathbf{v}\}.$$

Show that E_λ is G-invariant.

(iii). Let G_λ be the *generalized eigenspace* of A:

$$G_\lambda = \{\mathbf{v} \in V \mid (A - \lambda I)^n\mathbf{v} = \mathbf{0}\},$$

where $n = \dim V$. Show that G_λ is also G-invariant.

4.4 Consider the function $f(x, y) = x^2 + y^2 - x^4 - y^4$. Show this is invariant under the group D_4 and find its set $C(f)$ of critical points. Describe how the group acts on this set (i.e., determine the orbits and the orbit type for each orbit), and hence state the Burnside type of the action on $C(f)$.

4.5 Let V be a representation of G with $V^G = \{0\}$. Prove directly that if $f : V \to \mathbb{R}$ is an invariant function then it has a critical point at 0. [By directly, I mean do not use the Principle of Symmetric Criticality, but you may use its proof to inspire you.]

4.6 Find all the critical points of the D_3-invariant function $f(x, y) = \lambda(x^2 + y^2) + \frac{1}{3}x^3 - xy^2$. Relate these to the fixed point subspaces for different subgroups of D_3 (refer to Fig. 4.1).

4.7 For the system of 4 springs discussed in lectures (Example 4.12), study the critical points in the subspace $\mathrm{Fix}(K, \mathbb{R}^2)$, where $K = \langle r_{\pi/4}\rangle$.

4.8 Let G act on a set X, and let Ω be the set of all functions $f : X \to \mathbb{R}$. Show that the following formula defines an action of G on Ω:

$$(g \cdot \phi)(x) = \phi(g^{-1}x), \quad \text{for } \phi \in \Omega,\ g \in G,\ x \in X.$$

In other words, $g \cdot \phi = \phi \circ g^{-1}$.
[Note: The inverse here should be reminiscent of the action by right multiplication of a group on itself, from Chapter 1 (§1.3) which also involves an inverse.]

4.9 Suppose V, W are representations of a group G. Let $\phi_j : V \to W$ be two equivariant maps, and let $f_j : V \to \mathbb{R}$ be two invariant functions ($j = 1, 2$). Show that the map $\psi : V \to W$ given by

$$\psi(\mathbf{v}) = f_1(\mathbf{v})\,\phi_1(\mathbf{v}) + f_2(\mathbf{v})\,\phi_2(\mathbf{v})$$

is equivariant.

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4.10 Find all homomorphisms of the cyclic group $\mathbb{Z}_4$ to the cyclic group $\mathbb{Z}_6$. [Hint: If H is a cyclic group generated by a, and $\phi: H \to G$ a homomorphism, then ϕ is entirely determined by knowing $\phi(a)$.]