Chapter 1

Group actions

Groups manifest themselves by *acting* on sets, so displaying the symmetry of the set in question. This chapter describes many of the basic properties of group actions, introducing the language used and some of the most important general theorems. Before discussing group actions, we begin with a brief reminder about permutations.

1.1 Permutations

Consider the set $\{1,2,\dots,N\}$ (or any other set of N objects). A ***permutation*** of this set is a bijective map $\sigma:\{1,\dots,N\}\to\{1,\dots,N\}$. If we compose two such maps together, first applying τ and then σ, we get another permutation denoted $\sigma\circ\tau$, or as is more traditional in this context, simply $\sigma\tau$. It is the permutation $x\mapsto\sigma(\tau(x))$. The set of all permutations of $\{1,\dots,N\}$ equipped with this composition law form a group denoted by S_N which is called the ***symmetric group on N objects*** or also *permutation group*. Its order is $|S_N| = N!$. A permutation σ can be written in the two-line notation

$$\sigma=\begin{pmatrix}1 & 2 & \cdots & N\\ \sigma(1) & \sigma(2) & \cdots & \sigma(N)\end{pmatrix}$$

The elements of $\{1,\dots,N\}$ are listed in the first row and for each one, its image under σ is written below it in the second row. For example, if $N=5$, define the two permutations σ and τ by

$$\sigma=\begin{pmatrix}1&2&3&4&5\\5&4&3&2&1\end{pmatrix},\qquad \tau=\begin{pmatrix}1&2&3&4&5\\3&2&4&1&5\end{pmatrix}.$$

This means for example, $\sigma(2)=4$ and $\tau(1)=2$.

The composition $\sigma\tau$ (meaning first apply τ and then σ) is the permutation

$$\sigma\tau=\begin{pmatrix}1&2&3&4&5\\3&4&2&5&1\end{pmatrix}.$$

A permutation can also be written in the disjoint cycle notation. Starting from any particular $x\in\{1,\dots,N\}$, one writes the sequence $(x,\sigma(x),\sigma(\sigma(x)),\dots)$ of successive images under σ. When the image returns to x, a cycle is completed. One continues by choosing a new element y outside of the previous cycle and repeating the cycle construction. In this notation, our example above reads $\sigma = (1\,5)(2\,4)$ (or $\sigma = (1\,5)(2\,4)(3)$ but the (3) is redundant) and $\tau = (1\,3\,4)$. Then, $\sigma\tau = (1\,3\,2\,4\,5)$ (complete cycle) with inverse $(\sigma\tau)^{-1} = (5\,4\,2\,3\,1)$.

More generally, if X is any finite set, we denote by

$$\mathsf{Sym}(X) = \text{ the group of permutations of the elements of } X.$$

If $|X| = N$, then there is an isomorphism (indeed many isomorphisms) between $\mathsf{Sym}(X)$ and S_N. For example the vertices A, B, C of the triangle in Figure 1.1 are permuted by the reflections shown, giving permutations of the set $\{A, B, C\}$ (see the example below).

Finally, recall that if a permutation can be written as a product of r transpositions then one says it has ***sign*** equal to $(-1)^r$ (also called the *signature* or *parity*). It follows from this definition, and the fact that the transpositions generate the group) that the sign defines a homomorphism $\mathrm{sgn} : S_n \to \mathbb{Z}_2$. Note that the sign of a cycle of length ℓ is $(-1)^{\ell-1}$: for example $(1\,2\,3\,4) = (1\,4)(1\,3)(1\,2)$ so has sign $(-1)^3 = -1$.

1.2 Group actions

Now we turn to actions of groups; group actions is the mathematical language required for studying symmetry. We begin with an example and then give the general definition.

Notation Throughout this course, we will use the notation R_θ to mean a rotation of the plane about the origin through an angle θ in the anticlockwise direction, and r_α to mean the reflection through the line $y = x\tan\alpha$ (which subtends an angle α with the positive x-axis). Note that $R_{\theta+2\pi} = R_\theta$ and $r_\alpha = r_{\alpha+\pi}$.

Example 1.1 (the Dihedral group D_3). In this example, for brevity, we denote $\alpha = \frac{2\pi}{3}$ and $\beta = -\frac{2\pi}{3}$. Consider the so-called dihedral group of order 6,

$$G = \mathsf{D}_3 = \{e, R_\alpha, R_\beta, r_0, r_\alpha, r_\beta\}$$

of rotations and reflections in the plane. Multiplication in the group is by composition, so for example $r_\beta r_\alpha$ means $r_\beta \circ r_\alpha$ (do r_α first then r_β), and this is the rotation R_β through β. By looking at the effect on the plane one can find the multiplication

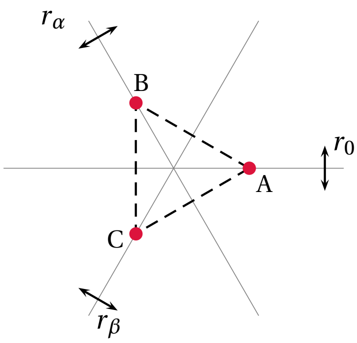


FIGURE 1.1: Symmetries of the equilateral triangle, with $\alpha = 2\pi/3$ and $\beta = 4\pi/3$

table:

$\boxed{D_3}$	e	R_α	R_β	r_0	r_α	r_β
e	e	R_α	R_β	r_0	r_α	r_β
R_α	R_α	R_β	e	r_β	r_0	r_α
R_β	R_β	e	R_α	r_α	r_β	r_0
r_0	r_0	r_α	r_β	e	R_α	R_β
r_α	r_α	r_β	r_0	R_β	e	R_α
r_β	r_β	r_0	r_α	R_α	R_β	e

$$\alpha = \tfrac{2\pi}{3}, \quad \beta = -\tfrac{2\pi}{3}$$

Each element of G permutes the elements of $V := \{A, B, C\}$ (the vertices of the equilateral triangle—see Fig. 1.1). This gives a map $\rho : \mathrm{D}_3 \to \mathrm{Sym}(V)$ defined by

$$\rho(e) = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}, \quad \rho(R_\beta) = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}, \quad \rho(R_\alpha) = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix},$$

$$\rho(r_0) = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}, \quad \rho(r_\alpha) = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}, \quad \rho(r_\beta) = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}.$$

This is clearly a homomorphism, that is, $\rho(gh) = \rho(g)\rho(h)$, because the multiplication operation of both groups is composition. For example,

$$\rho(r_\beta)\rho(r_\alpha) = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} = \rho(R_\beta) = \rho(r_\beta r_\alpha).$$

(In fact, in this case, this homomorphism is an isomorphism, but that is not usually the case.)

Definition 1.2. An ***action*** of a group G on a set X is a homomorphism

$$\rho : G \to \mathrm{Sym}(X).$$

One sometimes writes $G \circlearrowright X$ to mean that G acts on X. ✯

Recall that a map ϕ between two groups is a homomorphism if, for all g, h in the group,

$$\phi(gh) = \phi(g)\phi(h).$$

See the appendix (page A.8) for more details.

Example (The dihedral group D_3 continued). Let us extend the above example, with the same group but more points in the plane (Fig. 1.2).

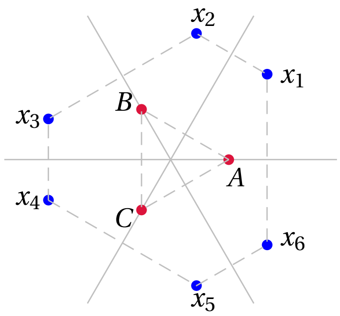


FIGURE 1.2: A set of 9 points with symmetry D_3. The 9 points form 2 orbits: an equilateral triangle and a semiregular hexagon. (The dashed lines are only to guide the eye.)

The dihedral group D_3 now acts on the set $X := \{A, B, C, x_1, x_2, x_3, x_4, x_5, x_6\}$. Then, there is a homomorphism $\rho : D_3 \to \mathrm{Sym}(X) \simeq S_9$. For instance, the image under ρ of the reflection r_0 is given by the permutation

$$\rho(r_0) = \begin{pmatrix} A & B & C & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ A & C & B & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 \end{pmatrix},$$

or in product of cycles notation, $\rho(r_0) = (B\,C)(1\,6)(2\,5)(3\,4)$. Since the groups D_3 and S_9 have different orders, ρ cannot be an isomorphism. We write $\rho(r_0)(x_3)$ (or more simply $\rho(r_0)x_3$) to mean the point x_3 is moved to when we apply the reflection r_0 to it, which here is the point x_4.

Sometimes one writes simply $g \cdot x$ in place of $\rho(g)x$.

Definition 1.3. If G acts on X, the ***orbit*** of a point $x \in X$ is the subset of X defined by

$$G \cdot x := \{\rho(g)x \mid g \in G\}.$$

✯

In the example in Fig. 1.2, $G \cdot A = \{A, B, C\}$ while $G \cdot x_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Therefore this action of D_3 has two orbits.

Definition 1.4. If G acts on X and $x \in X$, the ***stabilizer*** of x is the subset of G

$$G_x := \{g \in G \mid \rho(g)x = x\}.$$

(Stabilizer subgroups are also called *isotropy subgroups*.) If $\rho(g)x = x$ we say that g ***fixes*** x. ☆

In the example above, $G_A = \{e, r_0\}$, $G_B = \{e, r_\beta\}$, and $G_{x_1} = \{e\} = \mathbb{1}$.

Recall (see the appendix) that if G is a group, a non-empty subset $H \subset G$ is a *subgroup* if it is closed under the group operations of G; that is,

$$h \in H \implies h^{-1} \in H, \quad \text{and} \quad h_1, h_2 \in H \implies h_1 h_2 \in H.$$

This is called the *subgroup criterion* (note that together they imply $e \in H$). We write $H \le G$ as a shorthand to mean H is a subgroup of G.

The first fundamental property of group actions is the following.

Proposition 1.5. *Let G act on X. For each $x \in X$, the stabilizer G_x is a subgroup of G.*

Proof: First, by the homomorphism property of an action, the identity element $e \in G$ maps to the identity permutation, which fixes every element of X, so $e \in G_x$. Let $g \in G_x$. By definition, $\rho(g)x = x$. Then, since an action is a homomorphism,

$$\begin{aligned} \rho(g^{-1})x &= \rho(g^{-1})(\rho(g)x) \\ &= \rho(g^{-1}g)x \\ &= \rho(e)x = x. \end{aligned}$$

That is, $\rho(g^{-1})x = x$ and therefore, $g^{-1} \in G_x$. Finally, let $g_1, g_2 \in G_x$. Since $\rho(g_1 g_2)x = \rho(g_1)(\rho(g_2)x) = \rho(g_1)x = x$, we deduce $g_1 g_2 \in G_x$. Therefore, G_x is closed under the group operations of G which means it's a subgroup. ❐

Proposition 1.6. *Elements of X that lie on the same orbit have conjugate stabilizers; in particular, if $y = \rho(k)x$, then $G_y = kG_x k^{-1}$.*

Proof: Let $k \in G$ and let $y = \rho(k)x$. If $g \in G_x$, we get

$$\rho(kgk^{-1})y = \rho(kg)(\rho(k^{-1})y) = \rho(kg)x = \rho(k)(\rho(g)x) = \rho(k)x = y.$$

Therefore, $kG_x k^{-1} \subset G_y$. Conversely, if $h \in G_y$, then $\rho(k)x = y = \rho(h)y = \rho(hk)x$. Thus, $k^{-1}hk \in G_x$ which implies that $G_y \subset kG_x k^{-1}$. Therefore, $G_y = kG_x k^{-1}$. ❐

An alternative proof, which some may prefer, is as follows:

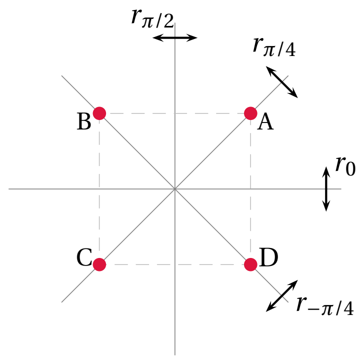


FIGURE 1.3: Symmetries of the square

Proof: Let $y = \rho(k)x$ as in the statement. Then (and here we're using the short-hand notation $g \cdot x$ instead of $\rho(g)x$),

$$\begin{aligned} G_y &= \{g \in G \mid g \cdot y = y\} \\ &= \{g \in G \mid g \cdot (k \cdot x) = k \cdot x\} \\ &= \{g \in G \mid (gk) \cdot x = k \cdot x\} \\ &= \{g \in G \mid (k^{-1}gk) \cdot x = x\} \\ &= \{khk^{-1} \in G \mid h \cdot x = x\} \quad \text{(putting } h = k^{-1}gk) \\ &= k\{h \in G \mid h \in G_x\}k^{-1} \\ &= kG_xk^{-1} \end{aligned}$$

as required. ❐

Example (the dihedral group D_4) See Figure 1.3. One can deduce from this action that the elements $r_{\pi/4}$ and $r_{-\pi/4}$ are conjugate elements of D_4: this is because the stabilizer $G_A = \{I, r_{\pi/4}\}$ and the stabilizer $G_B = \{I, r_{-\pi/4}\}$. Since the points A and B lie in the same orbit (because $B = R_{\pi/2} \cdot A$) so the stabilizers are conjugate. More specifically, $G_B = R_{\pi/2} G_A R_{\pi/2}^{-1}$.

On the other hand, r_0 and $r_{\pi/4}$ are *not* conjugate (see Problem 1.10).

Two of the subgroups of D_4 are

$$D_2 = \{I, r_0, r_{\pi/2}, R_\pi\}, \quad \text{and} \quad D_2' = \{I, r_{\pi/4}, r_{-\pi/4}, R_\pi\}.$$

Although both are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, they are not conjugate subgroups of D_4: one way to see this is that r_0 is not conjugate to any element of D_2'.

Reminder from group theory. Let $H \le G$. Then, for $g \in G$, the set

$$gH := \{gh \mid h \in H\}$$

is called a ***left coset*** of H in G. Similarly,

$$Hg := \{hg \mid h \in H\}$$

is a ***right coset*** of H in G. See the appendix for more details, and particularly the statement and proof of Lagrange's theorem (see p. A.6).

Let G act on X and for any pair of elements $x, y \in X$ consider the following subset of G,

$$G_{x,y} := \{g \in G \mid \rho(g)x = y\}.$$

In words, this set consists of those elements of the group that send x onto y. Of course, if x and y are not in the same orbit, then $G_{x,y}$ is empty.

Proposition 1.7. *If $y = \rho(h)x$ for some $h \in G$, then $G_{x,y} = hG_x = G_y h$; that is, $G_{x,y}$ is a left coset of the stabilizer G_x and a right coset of the stabilizer G_y.*

Proof: We show that $G_{x,y} = hG_x$ and leave the other to the reader. Now,

$$\begin{aligned} hG_x &= \{hg \mid g \in G_x\} \\ &= \{hg \mid \rho(g)x = x\} \quad \text{(definition of } G_x) \\ &= \{hg \mid \rho(h)\rho(g)x = \rho(h)x\} \quad \text{(acting by } \rho(h)) \\ &= \{k \in G \mid \rho(k)x = y\} \quad \text{(putting } k = hg\text{, and } y = h\cdot x) \\ &= G_{x,y}, \end{aligned}$$

as required (we used the homomorphism property: $\rho(h)\rho(g) = \rho(hg)$). ❒

The following theorem has many applications to counting things.

Theorem 1.8 (Orbit-Stabilizer theorem). *Suppose G acts on X and let $x \in X$. Then, the number of points in the orbit of x is*

$$|G\cdot x| = |G| \,/\, |G_x|.$$

Proof: To prove this, we define a map ϕ from the orbit $G\cdot x$ to the set of left cosets of G_x which we show is a bijection. The map in question is defined by

$$\begin{aligned} \phi : G\cdot x &\longrightarrow G/G_x \\ y &\longmapsto G_{x,y}. \end{aligned}$$

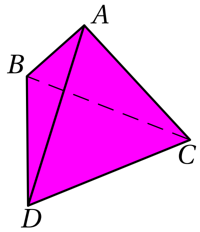
By the proposition above, this is indeed a left coset of G_x.

Now we show ϕ is a bijection. To see it is surjective we need to show that each left coset hG_x is equal to $\phi(y)$ for some $y \in G\cdot x$. But $\phi(h\cdot x) = hG_x$ by the proposition above, so putting $y = h\cdot x$ shows ϕ is indeed surjective. For the injectivity of ϕ, suppose y_1 and y_2 are two distinct points in the orbit. Then, by its very definition, $G_{x,y_1} \neq G_{x,y_2}$, so that $\phi(y_1) \neq \phi(y_2)$.

It follows that $|G\cdot x| = |G/G_x| = |G|/|G_x|$ (see Lagrange's theorem). ❒

For instance, for the action depicted in Figure 1.1, the group is $G = \mathrm{D}_3$ which has order $|\mathrm{D}_3| = 6$. The stabilizer subgroup $G_A = \{e, r_0\}$ has order $|G_A| = 2$ and thus the orbit $G\cdot A$ has three elements (namely, A, B and C).

Example 1.9 (Order of tetrahedral group $\mathbb{T}$). This theorem can also be used to determine the order of a group, as this example shows. Consider a regular tetrahedron, and let $\mathbb{T}$ denote its group of rotational symmetries: we want to determine the order of $\mathbb{T}$. Now, the tetrahedron has 4 vertices, call them A, B, C, D, and of course any symmetry of the tetrahedron permutes these vertices. Now select one of the vertices, say A. The orbit of A is clearly $\{A, B, C, D\}$, since one can rotate A to any of the other 3 vertices; that is, $|\mathbb{T} \cdot A| = 4$. We now need to find the stabilizer of A. Rotations fixing A must rotate the triangular face BCD opposite A, and there are 3 such possible rotations (including the identity). That is, the stabilizer $\mathbb{T}_A$ has order 3. It therefore follows from the orbit-stabilizer theorem that $|\mathbb{T}| = |\mathbb{T}_A| \times |\mathbb{T} \cdot A| = 3 \times 4 = 12$. See Problem 1.5 for a similar example.



It is useful to distinguish certain types of action:

Definition 1.10. Let $\rho : G \to \mathrm{Sym}(X)$ be an action of G on X. We say the action is,

- ***transitive*** if for every $x, y \in X$, there is a $g \in G$ such that $\rho(g)x = y$;
- ***free*** if for every $x \in X$, the isotropy group is trivial: $G_x = \mathbb{1}$;
- ***effective*** if for any $g \in G, g \neq e$, there exists $x \in X$ such that $\rho(g)x \neq x$.

☆

Examples 1.11. (1). **Triangle example (Fig. 1.1)** The action of D_3 on $V = \{A, B, C\}$ is not free since, for instance, $G_A = \{e, r_0\} \neq \mathbb{1}$. Moreover, this action is transitive. Indeed, any element of V can be moved to any other using a reflection. It is also effective because any element other that the identity moves at least one element).

(2). **Triangle example with six additional points (Figure 1.2)** Here D_3 acts on $\{A, B, C, x_1, \dots, x_6\}$. This action is not free since again $G_A \neq \mathbb{1}$. It is not transitive as there is no $g \in D_3$ such that $g \cdot A = x_1$ say. Indeed, we already know that this action has two orbits. It is an effective action as above.

(3). **Square example**. The action of D_4 on $\{A, B, C, D\}$ is not free, but is transitive and effective. The action—call it ρ—of D_4 on the diagonals of the square $\{AC, BD\}$ is also transitive. It is not effective since both diagonals are fixed by R_π. So $R_\pi \in \ker(\rho)$ because the resulting permutation of the diagonals is,

$$\rho(R_\pi) = \begin{pmatrix} AC & BD \\ AC & BD \end{pmatrix}.$$

In fact, $\ker(\rho) = \{e, R_\pi, r_{\pi/4}, r_{-\pi/4}\}$ which is a subgroup of order 4. Finally, since it is not effective, it is not free.

To end this section on general properties of actions, we define the notion of when two actions are 'equivalent', or 'isomorphic'. The first notion (that of equivariant maps) is one we will be using many times.

Definition 1.12. Suppose G acts on two sets X and Y by $\rho_X : G \to \mathsf{Sym}(X)$ and $\rho_Y : G \to \mathsf{Sym}(Y)$.

(1). A map $\phi : X \to Y$ is ***equivariant*** if it satisfies $\phi(\rho_X(g)(x)) = \rho_Y(\phi(x))$ for every $g \in G$ and every $x \in X$. More briefly, this condition can be written

$$\phi(g \cdot x) = g \cdot \phi(x). \tag{1.1}$$

(2). The actions on X and Y are said to be ***isomorphic*** if there is a bijection $\phi : X \to Y$ which is also equivariant.

This equivariance property can be illustrated by the diagram on the right: if you start in the top left X and proceed to the bottom right Y by either path you get the same answer. (Such a diagram is called a *commutative diagram*.)

$$\begin{tikzcd} X \arrow[r, "\rho_X(g)"'] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "\rho_Y(g)"] & Y \end{tikzcd}$$

Example 1.13. Let V denote the set of vertices of an equilateral triangle (as in Fig. 1.1) and E the set of its edges. The group D_3 acts on both sets, and these two actions are isomorphic. Indeed, the map $\phi : V \to E$ given by $\phi(A) = BC$, $\phi(B) = AC$ and $\phi(C) = AB$ is equivariant and bijective: checking this is left as an exercise (see Problem 1.14).

1.3 Groups acting on themselves

There are three general ways in which a group acts on itself (that is where $X = G$).

(1). The group G acts on itself by ***left translation***[1]. This action is given by the homomorphism

$$\lambda : G \to \mathsf{Sym}(G)$$

where $\lambda(g)$ is the permutation $k \mapsto gk$. Let us show that λ defines an action: that is that $\lambda(gh) = \lambda(g)\lambda(h)$. To see this, pick an element $k \in G$ and compute

$$\lambda(gh)(k) = (gh)k = g(hk) = \lambda(g)(hk) = \lambda(g)\lambda(h)(k).$$

Since k was arbitrary, it follows that indeed $\lambda(gh) = \lambda(g)\lambda(h)$. For instance, for the group D_3,

$$\lambda(R_\alpha) = \begin{pmatrix} e & R_\alpha & R_\beta & r_0 & r_\alpha & r_\beta \\ R_\alpha & R_\beta & e & r_\beta & r_0 & r_\alpha \end{pmatrix}.$$

[1] sometimes called ***left multiplication***

Notice that $\lambda(g)$ is the permutation of G given by the row of the multiplication table corresponding to g. This action λ is always free: $\lambda(g)(k) = k \Leftrightarrow gk = k \Leftrightarrow g = e$. It is also transitive since, for any $h, k \in G$, we can choose $g = kh^{-1}$ to get $\lambda(g)(h) = kh^{-1}h = k$. It is also effective since it is free.

(2). A group also acts on itself by ***right translation***. This is given by a homomorphism $\rho : G \to \mathsf{Sym}(G)$, where $\rho(g)$ is the permutation $k \mapsto kg^{-1}$ Thus $\rho(g)$ is the permutation of G given by the column of g^{-1}. We have, for example,

$$\rho(R_\alpha) = \begin{pmatrix} e & R_\alpha & R_\beta & r_0 & r_\alpha & r_\beta \\ R_\beta & e & R_\alpha & r_\beta & r_0 & r_\alpha \end{pmatrix}.$$

Note that the second row is the $R_\alpha^{-1} = R_\beta$ column of the multiplication table on p. 1.3. This map satisfies $\rho(gh) = \rho(g)\rho(h)$, as should be checked (it's not a homomorphism in general if we used $k \mapsto kg$ instead of $k \mapsto kg^{-1}$). Like the action on the left, this is free, transitive and effective.

(3). The third action is perhaps the most important: the action by ***conjugation***. This action $\mu : G \to \mathsf{Sym}(G)$ is defined by $\mu(g)(k) = gkg^{-1}$. This action is neither free nor transitive; whether it is effective or not depends on the group. For instance, if the group is commutative this action is trivial (as $gkg^{-1} = gg^{-1}k = k$) and therefore in this case $\ker\mu = G$. We return to this later.

Exercise: Show ρ and μ are actions. Show moreover that for each g, $\mu(g)$ is an automorphism of G, that is, $\mu : G \to \mathrm{Aut}(G)$ (see Appendix: an automorphism is an isomorphism of G with itself).

1.4 Action on left cosets

Fix a subgroup $H \leq G$, and consider the collection of left cosets of H in G. Denote this set by G/H:

$$G/H := \{gH \mid g \in G\}.$$

In general, G/H is not a group; it is just a set. From Lagrange's theorem (or Fig. A.1), $|G/H| = |G|/|H|$.

The left action $\lambda : G \to \mathsf{Sym}(G)$ defined above induces an action of G on G/H, which we denote $\lambda_H : G \to \mathsf{Sym}(G/H)$; this is defined by

$$\lambda_H(g)(kH) = gkH.$$

Example (See Figure 1.1). Consider the subgroup of D_3 given by $H := \langle r_0 \rangle = \{e, r_0\}$. There are 3 left cosets, which are: $H = \{e, r_0\}$, $r_\alpha H = \{r_\alpha, R_{-2\pi/3}\} = R_{-2\pi/3}H$ and $r_\beta H = \{r_\beta, R_{2\pi/3}\} = R_{2\pi/3}H$. The quotient space is thus $G/H = \mathrm{D}_3/\langle r_0 \rangle = \{H, r_\alpha H, r_\beta H\}$. The action $\lambda_H : G \to \mathsf{Sym}(G/H)$ is given by

$$\lambda_H(R_{2\pi/3}) = \begin{pmatrix} H & r_\alpha H & r_\beta H \\ r_\beta H & H & r_\alpha H \end{pmatrix}$$

and

$$\lambda_H(r_0) = \begin{pmatrix} H & r_\alpha H & r_\beta H \\ H & r_\beta H & r_\alpha H \end{pmatrix}$$

because, for example, $r_0 r_\alpha = R_{2\pi/3}$ so $r_0 r_\alpha H = R_{2\pi/3} H = r_\beta H$.

You may notice that this action of D_3 is very similar to the action of D_3 on the triangle (Example 1.1). With the following theorem we make that precise and see it is a general phenomenon.

Recall that the Orbit-Stabilizer theorem (Theorem 1.8) tells us that the number of elements in $G \cdot x$ is the same as the number of elements in G/G_x. The following theorem tells us more: that the two actions of G are isomorphic. This is a fundamental fact: it shows that the action of G on any group orbit is encoded in the action on the set of left cosets G/H for an appropriate choice of H.

Theorem 1.14. *Suppose G acts transitively on X (i.e., X is a single orbit), and let $x \in X$. Then the action of G on X is isomorphic to the action λ_H of G on G/H for $H = G_x$.*

Proof: Recall the map $\phi : G \cdot x \to G/G_x$ defined in the proof of the Orbit-Stabilizer theorem:

$$\phi(y) = G_{x,y}.$$

If $y = k \cdot x$ then we saw (Proposition 1.7) $\phi(y) = kG_x$. We proved in Theorem 1.8 that ϕ is a bijection, so there just remains to show it is equivariant: that is $\phi(g \cdot y) = g\phi(y)$ (for all $y \in X$ and all $g \in G$).

Choose $y \in$X, and let k be such that $y = k \cdot x$, and let $g \in G$. Then,

$$\begin{aligned} \phi(g \cdot y) &= \phi(gk \cdot x) \\ &= (gk)G_x \\ &= g(kG_x) \\ &= g\phi(y). \end{aligned}$$

Since $y \in X$ and $g \in G$ are arbitrary, the theorem is proved. ❒

Let us add a second equilateral triangle to the action of D_3 depicted in Figure 1.2, for example with vertices opposite to those of the first triangle, and an extra point at the origin, see Figure 1.4. There are now four orbits in total; namely, the origin, two equilateral triangles and one semi-regular hexagon:

$$\{O\}, \quad \{A, B, C\}, \quad \{A', B', C'\}, \quad \text{and} \quad \{x_1, x_2, \ldots, x_6\}.$$

The group D_3 would act on the two equilateral triangles in a similar way, while the action is quite different on the hexagon and the origin. The definition of 'orbit type' given below makes this precise.

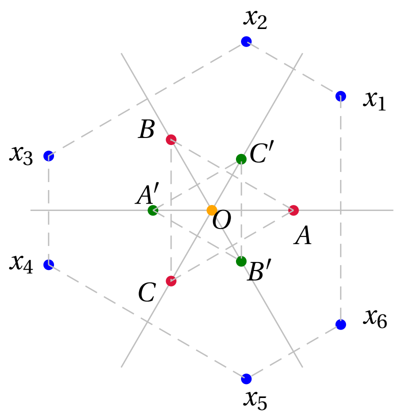


FIGURE 1.4

1.5 Orbit type

The theorem above suggests we can classify orbits of a given group action by their stabilizer, since the action on a given orbit is determined by the stabilizer of any point in the orbit. However, different points in the orbit might have different stabilizers so the suggestion is not sensible as it stands.

However, all is not lost: recall from Proposition 1.6 that points in the same orbit have conjugate stabilizers. For example, in the figure,

$$G_A = \langle r_0 \rangle, \quad \text{and} \quad G_B = \langle r_{-\pi/3} \rangle,$$

and r_0 and $r_{-\pi/3}$ (and $r_{\pi/3}$) are conjugate in D_3 (indeed, $r_{-\pi/3} = R_{2\pi/3}^{-1} r_0 R_{2\pi/3}$). Thus, associated to each orbit is the *conjugacy class* of the stabilizers. This motivates the following definition.

Definition 1.15. Suppose G acts on X. The ***orbit type*** of x, or of its orbit $G \cdot x$, is defined to be the conjugacy class of the stabilizer of x. ✯

In the example shown in Fig. 1.4 above, the points A, B, C, A', B', C' all have stabilizer conjugate to $\mathsf{D}_1 = \langle r_0 \rangle$. On the other hand O has stabilizer D_3 which is not conjugate to D_1, and each of the vertices of the semi-regular hexagon has stabilizer $\mathbb{1}$.

Notation If $H < G$ then we write (H) for the collection of subgroups of G that are conjugate to H. The orbit type of the triangle in Fig. 1.4 (in fact both triangles) is thus written (D_1), the orbit type of the origin is (D_3) and of the semi-regular hexagon is $(\mathbb{1})$.

Since in this example, there are one orbit of type (D_3), two of type (D_1) and one of type $(\mathbb{1})$, we can represent the action of D_3 on the set of 13 points as,

$$(\mathsf{D}_3) + 2(\mathsf{D}_1) + (\mathbb{1}),$$

and this information tells us exactly ow the group is acting on the set. This expression is known as the ***Burnside type*** of the action, after the group theorist William Burnside who developed many of these ideas around the end of the 19th century. The formal definition in general is as follows.

Definition 1.16. Consider a finite group G and let $H_1,\dots,H_k$ be a collection of subgroups such that every subgroup of G is conjugate to exactly one of these H_i. Now suppose G acts on a finite set S. We can write S as a disjoint union of orbits and each of these orbits has orbit type equal to one of the (H_i). Suppose the number of orbits of type (H_i) is n_i. Then we say the ***Burnside type*** of the G action on S is,

$$\mathcal{B}(S) = n_1(H_1) + n_2(H_2) + \cdots n_k(H_k).$$

Here the n_i are non-negative integers. ✯

Notice that the total number of orbits in the above expression is

$$n_1 + n_2 + \cdots + n_k.$$

One denotes the ***set of orbits*** of an action of G on X by X/G, and it is often called the ***quotient*** of X by G: more on this in the next section.

Note also that if S is the disjoint union of two sets S_1 and S_2, and G acts on each of these, then the Burnside type of S is the sum of the Burnside types of S_1 and S_2.

Example 1.17. Consider $G = \mathsf{D}_4$ acting on the square with vertices A, B, C, D, as in Figure 1.3. Let $V = \{A, B, C, D\}$ be this set of vertices and $E = \{AB, BC, CD, DA\}$ be its set of edges. Let $S = V \cup E$. The sets V and E are both orbits of D_4 with 4 elements. The stabilizer of the point $A \in V$ is $\mathsf{D}_1' = \langle r_{\pi/4}\rangle$, while the stabilizer of the edge AB is $\mathsf{D}_1 = \langle r_0\rangle$. Thus the Burnside type of S is

$$\mathcal{B}(S) = 1(\mathsf{D}_1) + 1(\mathsf{D}_1').$$

(It was pointed out earlier that D_1 and D_1' are not conjugate subgroups of D_4; see also Problem 1.10.)

1.6 Counting orbits

Suppose G acts on a set X. It is natural to ask how many orbits there are. For example, in Figure 1.2 there are two orbits and in Figure 1.4 (on p.1.12) there are 4. There is a formula expressing the number of orbits in terms of the number of points fixed by the different elements of G. It is often called the Burnside Lemma, although it can also be attributed[2] to Cauchy and to Frobenius.

For each $g \in G$, denote by X^g the subset of elements of X that are fixed by g; that is,

$$X^g \;=\; \left\{x \in X \mid \rho(g)x = x\right\}.$$

[2] Augustin Cauchy lived 1789–1857, so was active 150 years before Burnside; Frobenius on the other hand was a contemporary of Burnside

Theorem 1.18 (Orbit counting theorem, or Burnside Lemma). *Let a finite group G act on a finite set X. Then the number of orbits is*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}\left|X^g\right|.$$

Let us test this theorem on the action of D_3 on the 13 points shown in Fig. 1.4. We have the following table of numbers of fixed points

e	$R_{2\pi/3}$	$R_{4\pi/3}$	r_0	$r_{\pi/3}$	$r_{-\pi/3}$
13	1	1	3	3	3

The order of the group is 6, so the number of orbits is

$$|X/G| = \frac{1}{6}(13+1+1+3+3+3) = 4.$$

That is, as we already know, this action has 4 orbits. Note that conjugate elements of the group have the same number of fixed points (see Problem 1.4).

Proof: This is an exercise in counting something in two different ways. The 'something' is the set S of[3] 'stabilizing pairs',

$$S = \{(g,x)\in G\times X \mid \rho(g)x = x\},$$

where ρ is the action of G on X. We proceed to find the cardinality of S in two different ways.

First, for each $g\in G$, $(g,x)\in S$ iff $x\in X^g$, and so

$$|S| = \sum_{g\in G}|X^g|. \tag{1.2}$$

On the other hand, for each $x\in X$, there are $|G_x|$ elements g such that $(g,x)\in S$. Thus

$$|S| = \sum_{x\in X}|G_x| = \sum_{x\in X}\frac{|G|}{|G\cdot x|} = |G|\sum_{x\in X}\frac{1}{|G\cdot x|},$$

(the second equality is from the Orbit-Stabilizer theorem). Now we need to simplify the final expression, and to do this we divide X up into the orbits, say r of them: $\mathcal{O}_1,\ldots,\mathcal{O}_r$ (so $\mathcal{O}_j\in X/G$). Then we can rewrite the sum as,

$$\sum_{x\in X}\frac{1}{|G\cdot x|} = \sum_{\mathcal{O}\in X/G}\left(\sum_{x\in\mathcal{O}}\frac{1}{|\mathcal{O}|}\right).$$

Then for each orbit the sum gives $\sum_{x\in\mathcal{O}}\frac{1}{|\mathcal{O}|} = |\mathcal{O}|\frac{1}{|\mathcal{O}|} = 1$. Thus each orbit contributes 1 to the final sum, and therefore,

$$|S| = |G|\sum_{\mathcal{O}\in X/G}1 = |G||X/G|.$$

The theorem now follows by combining this with (1.2) above. ❐

[3]The reader may like to see what the set S would be for the action of D_3 on the set in Figure 1.2

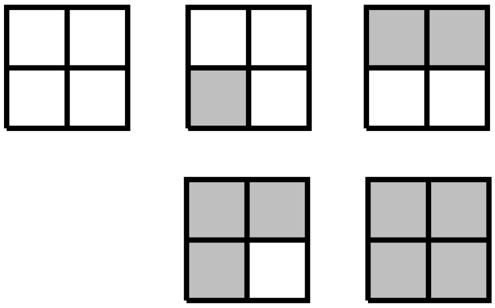


FIGURE 1.5: The 6 colourings of a 2×2 square with 2 colours

Example 1.19. How many essentially different ways are there of colouring a 2×2 square given 3 coloured paints. Here 'essentially different' means that we do not want to distinguish between 2 colourings that differ only be a rotation of the square.

Let us label the small squares A, B, C, D. If we did not use the term 'essentially' here, there would be 3^4 different possible colourings: 3 for square A, 3 for square B and so on. In terms of symmetry, we have an action of C_4 on the square, and hence on all the possible colourings, and the essentially different ones are those that lie in different orbits for this action. Thus, we want to know how many orbits there are under the action of the rotation group C_4 acting on the set X of 3^4 colourings.

We apply the orbit-counting theorem: so for each element of C_4 we need to know how many colourings are fixed by that element. For the identity $I \in \mathrm{C}_4$, every colouring is fixed, so $X^I = X$ and $|X^I| = 3^4$. A colouring fixed by $R_{\pi/2} \in \mathrm{C}_4$ must have all 4 small squares coloured the same, and therefore $|X^{R_{\pi/2}}| = 3$. For $R_{-\pi/2}$ the answer is the same. However for R_π, once we specify the colours of A and B, the other two are determined, and thus there are $3 \times 3 = 9$ possibilities. Thus $|X^{R_\pi}| = 9$. Consequently

$$|X/\mathrm{C}_4| = \frac{1}{4}(3^4 + 3 + 9 + 3) = 24.$$

It is left as an exercise to find all such 24 possible colourings. The analogous question with two colours gives $\frac{1}{4}(2^4 + 2 + 4 + 2) = 6$ as the answer, and the 6 possible colourings are shown in Figure 1.5.

1.7 Problems

†Questions at the end marked with a † are beyond the syllabus, and just for interest

1.1 Let $\mathbb{Z}_2 = \langle \kappa \rangle$ act on $\mathbb{R}$ by $\rho(\kappa)x = -x$. Find the stabilizer and orbit of each element $x \in \mathbb{R}$.

1.2 Suppose two groups G, H both act on a set X via homomorphisms ρ_G and ρ_H

respectively, in such a way that, for all $g \in G$, $h \in H$,

$$\rho_H(h) \circ \rho_G(g) = \rho_G(g) \circ \rho_H(h)$$

(one says the two actions *commute*). Show that this gives rise to an action $\overline{\rho}$ of the Cartesian product group $G \times H$ on X, defined by

$$\overline{\rho}(g,h)x = \rho_G(g) \circ \rho_H(h)x.$$

1.3 Suppose G is an abelian group and acts on a set X. Show that if $x, y \in X$ lie in the same orbit then their stabilizers are equal: (a) deduce this from Proposition 1.6, and (b) prove it directly.

1.4 Let G be a finite group acting on the finite set X. For $g \in G$ let X^g denote the subset of X consisting of those elements fixed by g; that is

$$X^g = \{x \in X \mid \rho(g)x = x\}.$$

Suppose g_1 is conjugate to g_2, say $g_2 = hg_1h^{-1}$. Show that $x \in X^{g_1} \iff \rho(h)x \in X^{g_2}$. Deduce that $|X^{g_1}| = |X^{g_2}|$.

1.5 Draw a picture of a cube, and let $\mathbb{O}$ denote the group of all rotations of the cube (called the *octahedral group*).
(a) Apply the orbit-stabilizer theorem to a face of the cube to find the order of $\mathbb{O}$.
(b) Now apply the same theorem to a vertex of the cube to check your answer.

1.6 For the group D_3 of symmetries of the equilateral triangle (see first example), write down the permutations of D_3 arising as $\lambda(R_{2\pi/3})$ and $\rho(R_{2\pi/3})$, and finally of conjugation by $R_{2\pi/3}$.

1.7 Consider the action of a group G on itself by conjugation. Use the Orbit-Stabilizer theorem to prove the *Class Formula*, which states that the number of elements of G that are conjugate to g is equal to $|G|/|C(g)|$, where $C(g)$ is the centralizer of g in G (see the appendix for centralizers).

1.8 Let $\rho : G \to \mathsf{Sym}(G)$ be the action by right multiplication defined in lectures: $\rho(g)(h) = hg^{-1}$. Show this is an action, but that in general the map $\rho' : G \to \mathsf{Sym}(G)$ given by $\rho'(g)(h) = hg$ is not an action.

1.9 Suppose G acts on a set X. (a) Show that the action is effective if and only if the homomorphism $\rho : G \to \mathsf{Sym}(X)$ is injective. (b) Show that an action is transitive if and only if there is only one orbit, X itself

1.10 Suppose G acts on X and let $x \in X$ and $H = G_x$. Suppose H' is conjugate to H. Show that there is a point $y \in X$ with stabilizer H'. Deduce from the action of D_4 on the vertices of a square that r_0 and $r_{\pi/4}$ are not conjugate in D_4.

1.11 Let X be a finite set, and denote by $\mathcal{P}(X)$ the *power set* of X (that is, the collection of all $2^{|X|}$ subsets of X). Now suppose a group G acts on X.

(i). Show that the following formula defines an action of G on $\mathcal{P}(X)$:

$$g \cdot S = \{\rho(g)x \mid x \in S\} \quad (g \in G,\ S \subset X).$$

(ii). Let $\kappa : \mathcal{P}(X) \to \mathcal{P}(X)$ be the 'complement map', $\kappa(S) = S' = X \setminus S$. Show that κ is a bijection and is equivariant.

(iii). For $k = 0, \ldots, |X|$ denote by $\mathcal{P}(X)_k$ the collection of those subsets of X with cardinality k. Use the map κ from above to show that the action of G on $\mathcal{P}(X)_k$ is isomorphic to the action on $\mathcal{P}(X)_{n-k}$, where $n = |X|$.

(iv). Show that together, G and $\mathbb{Z}_2 = \langle\kappa\rangle$ define an action of $G \times \mathbb{Z}_2$ on $\mathcal{P}(X)$ (see Problem 1.2 above).

1.12 Let H be a subgroup of K and K a subgroup of G; that is $H < K < G$. Show that the map

$$\pi : G/H \to G/K, \quad \pi(gH) = gK$$

is equivariant. (Here the actions of G on G/H and G/K are λ_H and λ_K, as defined in Section 1.4.)

1.13 Let H and K be two subgroups of a group G, and suppose $\psi : G/H \to G/K$ is a G-equivariant map for the actions λ_H and λ_K respectively.

(a) Show that ψ is surjective.

(b) Let $g_0 \in G$ be such that $\psi(H) = g_0K$. Show that $H < g_0Kg_0^{-1}$.

1.14 (a) Show the map ϕ given in Example 1.13 is equivariant, and hence deduce as claimed that the actions of D_3 on V and E are isomorphic.

(b) Now consider a square with vertices $V = \{A, B, C, D\}$ and edges $E = \{e, f, g, h\}$, and let D_4 act on this square. By considering the fixed points of the elements of the group show that the actions on V and E are *not* isomorphic.

1.15 Consider the action of the group D_4 on the set of 21 points shown in Fig. 1.6 below. (i) Determine its Burnside type. (ii) Verify the orbit counting theorem for this action of D_4.

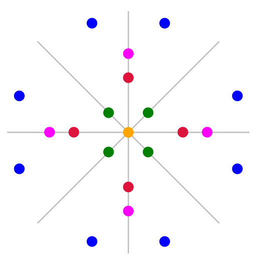


FIGURE 1.6

1.16 Suppose a group G acts on two sets X and Y. Consider the set $\mathrm{Map}(X, Y)$ of all maps from X to Y. Define an action ρ_M of G on this set of maps, by putting, for $\phi \in \mathrm{Map}(X, Y)$,

$$\rho_M(g)\phi = \rho_Y(g) \circ \phi \circ \rho_X(g)^{-1}$$

(note that $\rho_X(g^{-1}) = \rho_X(g)^{-1}$). Show first that ρ_M does indeed define an action on $\mathrm{Map}(X, Y)$, and second that $\phi \in \mathrm{Map}(X, Y)$ is fixed by all of G if and only if ϕ is equivariant.

1.17 Consider the right action of a subgroup K on a group G. Show that the orbits of this action are the left cosets of K, and deduce that the set of orbits is G/K (showing the notation of cosets is compatible with the notation of orbit space).

1.18 Consider a disc divided into 6 equal sectors. You have 3 colours at your disposal to colour the 6 segments. Find the number of distinct colourings there are, where (a) distinct means 'up to rotation', and (b) it means 'up to rotation and reflection'.

1.19† Suppose the group G acts on the set X and let H be a subgroup of G. Denote by X_H the subset $X_H = \{x \in X \mid G_x = H\}$, and suppose $x \in X_H$. Show that $g \cdot x \in X_H$ if and only if $g \in N_G(H)$ (the normalizer of H in G).

1.20† *Suppose H is a subgroup of G, and that H acts on a set X. It is natural to ask whether one can extend the action of H to one of G. In general the answer is no. However there is an extension of the set X which does allow this, defined as follows.*

Consider first the set $G \times X$, with an action of H given by

$$\sigma(h)(g, x) \ = \ (gh^{-1}, h \cdot x),$$

where $h \cdot x$ denotes the given action of H on X (alternatively, you can write $\rho_X(h)x$). Now define $Y = G \times_H X$ to be the quotient of $G \times X$ by this H-action. Thus, an element $[g, x] \in Y$ is an equivalence class,

$$[g, x] \ = \ \left\{(gh^{-1}, h \cdot x) \mid h \in H\right\} \ \subset \ G \times X.$$

(i). Show $[e, h \cdot x] = [h, x]$ (for all $h \in H$ and $x \in X$).

(ii). Show that the formula

$$\rho_Y(g)[g', x] = [gg', x]$$

defines a well-defined action ρ_Y of G on Y.

(iii). Show that the map $\phi : X/H \to Y/G$ defined by $\phi(H \cdot x) = G \cdot [e, x]$ is well-defined, and defines a bijection between X/H and Y/G.

1.21 † Let H and K be two subgroups of a group G. A subset of G of the form

$$HgK := \{hgk \mid h \in H,\ k \in K\} \subseteq G$$

is called a ***double coset***. The set of all such double cosets is denoted $H\backslash G/K$.

(i). The action λ_K of G on G/K (defined in §1.4) restricts on an action of H. Show that each double coset can be identified with an orbit of this action of H on G/K.

(ii). Let $G = S_3$, with H, K being the subgroups of order 2 given by

$$H = \langle (1\,2) \rangle, \quad K = \langle (1\,3) \rangle.$$

List the double cosets of S_3. [Hint: there are just 2 and they are not of the same size.]

(iii). Now let $G = S_4$, and let H, K be as above. How many double cosets are there?

