

## CHAPTER 5

# ROTATING FRAMES

**I**N CHAPTER 1 WE SAW the effect of changing inertial frames. However, it is also important to understand how the laws of mechanics appear if one is not in an inertial frame but in a frame that is uniformly rotating—if for no other reason than that we live on such a frame. Before studying rotating frames, we briefly consider the effect of two inertial frames, one rotated and translated relative to the other.

Indeed, we stressed at the beginning of the first chapter that mechanics takes place in Euclidean space, and that there is no preferred origin or axes. So as mathematicians, we need to be certain that the definitions we have made (such as kinetic energy) do not depend on the chosen frame, and that the vectors such as momentum and angular momentum transform as one would expect vectors to (due to a rotation of the axes).

---

### 5.1 CHANGE OF FRAME

---

In Linear Algebra courses, one learns how to change from one basis to another—we begin by recalling how this works. An extra feature now is that the origin can change too. Refer to Fig. 5.1, which is an illustration for 2-dimensional worlds (the  $z$ -axes are not shown). In fact the arguments below work in any number of dimensions.

First, recall a useful fact<sup>1</sup> about matrix multiplication. If  $R$  is an  $n \times n$  matrix, and  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$  (transpose as we want a column vector), then

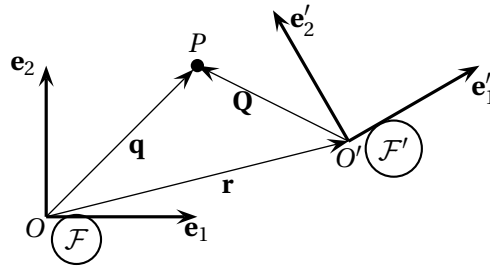
$$R\mathbf{x} = x_1\mathbf{R}_1 + x_2\mathbf{R}_2 + \cdots + x_n\mathbf{R}_n, \quad (5.1)$$

where  $\mathbf{R}_i$  is the  $i^{\text{th}}$  column of the matrix  $R$ . In particular,  $\mathbf{R}_i = R\mathbf{e}_i$  (for each  $i$ ).

Imagine we have two right-handed frames: frame  $\mathcal{F}$  with origin  $O$  and orthonormal basis vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ , and frame  $\mathcal{F}'$  with origin  $O'$  and orthonormal basis vectors  $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ . Eventually we will allow the second frame to be moving with respect to the first, but for now consider them as both stationary. The bases are related by a matrix as follows. Write each  $\mathbf{e}'_i$  as a column vector in terms of the  $\mathbf{e}_j$ , then place the columns together

---

<sup>1</sup>if you don't remember this, just check it for say a  $3 \times 3$  matrix—it's a useful exercise

FIGURE 5.1: Two frames, with  $\mathbf{q} = \mathbf{r} + R\mathbf{Q}$ 

in order to form a square matrix  $R$ . Then, by construction (see Eq. (5.1)),

$$\mathbf{e}'_i = R\mathbf{e}_i, \quad (\text{for } i = 1, 2, 3).$$

An equivalent way to construct  $R$  is that its entries are  $R_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$ . We will see below that because both bases are orthonormal,  $R$  is an orthogonal matrix, meaning that  $R^T R = I$ .

**EXAMPLE 5.1.** In 2 dimensions, the basis vectors will just be rotated one with respect to the other. If  $\theta$  is the angle of rotation (anti-clockwise, as usual), then  $\mathbf{e}'_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ , and  $\mathbf{e}'_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$  as is easy to see. Thus

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

You can check that  $R_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$  and that  $RR^T = I$ .

Now consider a point  $P$  in space. Its position vector in frame  $\mathcal{F}$  is say  $\mathbf{q} = (x, y, z)^T$ , and its position vector in frame  $\mathcal{F}'$  is  $\mathbf{Q} = (X, Y, Z)^T$  (transposes again as we want column vectors), see Fig. 5.1. These mean,

$$\overrightarrow{OP} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3, \quad \text{and} \quad \overrightarrow{O'P} = X\mathbf{e}'_1 + Y\mathbf{e}'_2 + Z\mathbf{e}'_3$$

To relate these, note first that  $\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P} = \mathbf{r} + \overrightarrow{O'P}$ , where we use  $\mathbf{r} = \overrightarrow{OO'}$  as in the figure, and that

$$\overrightarrow{O'P} = X\mathbf{e}'_1 + Y\mathbf{e}'_2 + Z\mathbf{e}'_3 = XR\mathbf{e}_1 + YR\mathbf{e}_2 + ZR\mathbf{e}_3 = X\mathbf{R}_1 + Y\mathbf{R}_2 + Z\mathbf{R}_3$$

(the columns of  $R$ ) so showing (see (5.1)) that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r} + R \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

or  $\mathbf{q} = \mathbf{r} + R\mathbf{Q}$ . We have thereby proved the following.

**PROPOSITION 5.2.** *The position vectors of a point  $P$  with respect to the two frames above are related by*

$$\mathbf{q} = R\mathbf{Q} + \mathbf{r}. \quad (5.2)$$

Now we show that  $R$  is orthogonal.

**LEMMA 5.3.** *The change of basis matrix  $R$  is orthogonal:  $R^T R = I$ . In particular, this implies that for any vectors  $\mathbf{u}, \mathbf{v}$  one has*

$$(R\mathbf{u}) \cdot (R\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}.$$

In particular, it follows that  $\|R\mathbf{u}\|^2 = (R\mathbf{u}) \cdot (R\mathbf{u}) = \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ . Thus orthogonal matrices preserve inner products and magnitudes of vectors.

**PROOF:** To see  $R$  is orthogonal, just use the fact that the bases are orthonormal:

$$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = \mathbf{R}_i \cdot \mathbf{R}_j = \sum_k R_{ki} R_{kj},$$

which is precisely the statement that  $R^T R = I$  (recall,  $\mathbf{R}_i$  is the  $i^{\text{th}}$  column of  $R$ ).

To see the statement about the scalar products, recall that  $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$  (where  $\mathbf{u}, \mathbf{v}$  are column vectors). Then

$$(R\mathbf{u}) \cdot (R\mathbf{v}) = \mathbf{u}^T R^T R \mathbf{v} = \mathbf{u}^T \mathbf{v} = \mathbf{u} \cdot \mathbf{v},$$

as required (using  $R^T R = I$  in the middle). □

**MECHANICS** Next we want to check that the basic mechanical definitions (such as kinetic energy and momentum) we have made are independent of the frame chosen, if they are not moving relative to one another. First consider kinetic energy of a moving particle of mass  $m$ . Write  $T$  and  $T'$  for the kinetic energies in the two frames. Differentiating (5.2) gives

$$\dot{\mathbf{q}} = R\dot{\mathbf{Q}}.$$

Substituting this into the kinetic energy:

$$T = \frac{1}{2} m \|\dot{\mathbf{q}}\|^2 = \frac{1}{2} m \|R\dot{\mathbf{Q}}\|^2 = \frac{1}{2} m \|\dot{\mathbf{Q}}\|^2 = T'.$$

(Here we used the lemma above.)

Similarly for momentum,

$$\mathbf{p} = m\dot{\mathbf{q}} = mR\dot{\mathbf{Q}} = R(m\dot{\mathbf{Q}}) = R\mathbf{P}$$

(writing  $\mathbf{p}$  for momentum in frame  $\mathcal{F}$  and  $\mathbf{P}$  for the momentum in frame  $\mathcal{F}'$ ). This is as one would expect, the momentum ‘rotates’ as any other vector does.

For the angular momentum the matter is a little more subtle, as the frames have different origins. Write  $\mathbf{L}$  for the angular momentum written in frame  $\mathcal{F}$  about its origin  $O$  and  $\mathbf{L}'$  for the angular momentum of the particle in frame  $\mathcal{F}'$  about its origin  $O'$ . Then,

$$\begin{aligned}\mathbf{L} &= \mathbf{q} \times \mathbf{p} \\ &= (R\mathbf{Q} + \mathbf{r}) \times (R\mathbf{P}) \\ &= (R\mathbf{Q}) \times (R\mathbf{P}) + \mathbf{r} \times (R\mathbf{P}) \\ &= R(\mathbf{Q} \times \mathbf{P}) + \mathbf{r} \times (R\mathbf{P}) \\ &= R\mathbf{L}' + \mathbf{r} \times (R\mathbf{P}).\end{aligned}$$

In particular if the origins coincide, so  $\mathbf{r} = 0$ , we have  $\mathbf{L} = R\mathbf{L}'$  as one would expect.

---

## 5.2 MOVING FRAMES

---

We now move to the more interesting case where the frame  $\mathcal{F}'$  is moving relative to the frame  $\mathcal{F}$ . If  $P$  is a moving particle then we still have, at each instant,

$$\mathbf{q} = R\mathbf{Q} + \mathbf{r}$$

but now each term depends on time  $t$ , so  $\mathbf{q}(t) = R(t)\mathbf{Q}(t) + \mathbf{r}(t)$ . We will refer to quantities measured in the frame  $\mathcal{F}$  as *absolute* and those measured in the frame  $\mathcal{F}'$  as *apparent*. Thus a moving particle has absolute velocity  $\dot{\mathbf{q}}$  and apparent velocity  $\dot{\mathbf{Q}}$ .

We will from now on concentrate on the more interesting effect of rotations, and suppose that the frames have the same origin, so  $\mathbf{r} = 0$ . It is straightforward to repeat this section with  $\mathbf{r} = \mathbf{r}(t)$  not being zero. So now we have the absolute and apparent position vectors of  $P$  are related by,

$$\mathbf{q}(t) = R(t)\mathbf{Q}(t).$$

To see how the absolute and apparent velocities are related, we differentiate this equation with respect to time. This gives

$$\dot{\mathbf{q}} = \dot{R}\mathbf{Q} + R\dot{\mathbf{Q}}.$$

We need to understand  $\dot{R}$ .

### 5.2A THE ANGULAR VELOCITY MATRIX

---

**DEFINITION 5.4.** Given  $R(t)$  define a matrix  $\Omega = \Omega(t)$  by

$$\Omega = R^{-1}\dot{R}.$$

This called the *angular velocity matrix*. Note that this is equivalent to  $\dot{R} = R\Omega$ . ✓

Note that since  $R$  is orthogonal, one also writes  $\Omega = R^T\dot{R}$ .

**EXAMPLE 5.5.** In the 2-dimensional case, with  $R$  given in Example 5.1, suppose  $\theta = \theta(t)$ . Then

$$\dot{R} = \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \dot{\theta}.$$

This gives

$$\Omega = R^T \dot{R} = \begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix}.$$

Notice that this matrix is skew-symmetric.

**LEMMA 5.6.** *The matrix  $\Omega$  is skew-symmetric:  $\Omega^T = -\Omega$ .*

**PROOF:**  $R(t)$  is always an orthogonal matrix, so  $R^T R \equiv I$ . Differentiating w.r.t.  $t$  gives

$$R^T \dot{R} + \dot{R}^T R = 0.$$

Therefore, for  $\Omega = R^T \dot{R}$  we have

$$\Omega^T = \dot{R}^T R = -R^T \dot{R} = -\Omega. \quad \square$$

The definition and the skew-symmetry of the angular velocity matrix does not depend on the dimension—it would be the same if we lived in  $\mathbb{R}^4$  (only it would be a  $4 \times 4$  matrix instead) or indeed  $\mathbb{R}^2$  as in the example above. In fact in 2 dimensions, any skew symmetric matrix is clearly of the form,

$$\Omega = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

The special feature of living in  $\mathbb{R}^3$  is that  $\Omega$  is  $3 \times 3$  and the set of skew-symmetric  $3 \times 3$  matrices is 3-dimensional (a coincidence; in  $\mathbb{R}^4$  the set of  $4 \times 4$  skew-symmetric matrices has dimension 6). This isomorphism between  $3 \times 3$  skew-symmetric matrices and vectors in  $\mathbb{R}^3$  is described in the following result, which is at the basis of the vector product (every time a vector product arises in an equation it is fundamentally because of this lemma).

**LEMMA 5.7.** *Let  $\Omega$  be a skew-symmetric  $3 \times 3$  matrix. Then there is a vector  $\boldsymbol{\omega} \in \mathbb{R}^3$  such that,*

$$\forall \mathbf{u} \in \mathbb{R}^3, \quad \Omega \mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}, \quad (5.3)$$

where  $\boldsymbol{\omega} \times \mathbf{u}$  is the vector product (or cross product) of the two vectors. This correspondence is given explicitly by the expression

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \mapsto \boldsymbol{\omega} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Moreover, if  $\Omega$  represents the instantaneous angular velocity of a motion, then the corresponding vector  $\boldsymbol{\omega}$  is parallel to the instantaneous axis of rotation.

**PROOF:** It is left to the reader to check that  $\Omega \mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$ . To see that  $\boldsymbol{\omega}$  is parallel to the instantaneous axis of rotation, it is enough (why?) to note that  $\Omega \boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\omega} = 0$ .  $\square$

## 5.2B VELOCITIES IN ROTATING FRAMES

Imagine living on a rotating disc world: a flat disc rotating about its axis with constant angular velocity  $\boldsymbol{\omega}$  (possibly resting on four elephants, which are standing on the back of a giant turtle?). Now drop a stone into a wide well. Will it fall vertically? Yes, according to an observer in an inertial frame. But an observer living in discworld will see the stone fall with a sideways component. How large is that component?

By differentiating  $\mathbf{q} = R\mathbf{Q}$  with respect to time we obtained (above):

$$\dot{\mathbf{q}} = R\dot{\mathbf{Q}} + \dot{R}\mathbf{Q}.$$

Substituting  $\dot{R} = R\Omega$  gives

$$\dot{\mathbf{q}} = R[\dot{\mathbf{Q}} + \Omega\mathbf{Q}]. \quad (5.4)$$

Using Lemma 5.7 above, this can be written

$$\dot{\mathbf{q}} = R[\boldsymbol{\omega} \times \mathbf{Q} + \dot{\mathbf{Q}}].$$

Let us rewrite this with  $\mathbf{v}$  as the absolute velocity and  $\mathbf{V}$  as the apparent velocity of the particle  $P$ . This gives

$$\mathbf{v} = R[\mathbf{V} + \boldsymbol{\omega} \times \mathbf{Q}]. \quad (5.5)$$

We see in particular that the absolute velocity is not just the transformation of the apparent velocity by the rotation  $R$ , there is another term involved due to the rotation of the frame  $\mathcal{F}'$ .

For example, an observer on a playground roundabout will observe a tree (so  $\mathbf{v} = \dot{\mathbf{q}} = 0$ ) as apparently moving with velocity

$$\mathbf{V} = -\boldsymbol{\omega} \times \mathbf{Q} = \mathbf{Q} \times \boldsymbol{\omega},$$

where  $\boldsymbol{\omega}$  is the angular velocity of the roundabout.

We are now in a position to answer the question above about life on discworld. If  $\mathbf{Q}$  is the radial position vector from the centre of the disc, then the apparent velocity the stone falls at is

$$\mathbf{V} = R^T\mathbf{v} - \boldsymbol{\omega} \times \mathbf{Q}.$$

The  $R^T\mathbf{v}$  term is the vertical absolute motion (the  $R^T$  just writes it in the moving basis) and the other term  $\boldsymbol{\omega} \times \mathbf{Q}$  is the extra term giving a sideways component.

## 5.2C ACCELERATION IN ROTATING FRAMES

To obtain an expression for the apparent acceleration in a rotating frame, we differentiate 5.4 again, to get

$$\ddot{\mathbf{q}} = R[\ddot{\mathbf{Q}} + \Omega\dot{\mathbf{Q}} + \dot{\Omega}\mathbf{Q}] + \dot{R}[\dot{\mathbf{Q}} + \Omega\mathbf{Q}].$$

Substituting again  $\dot{R} = R\Omega$  and rearranging gives,

$$\ddot{\mathbf{q}} = R(\ddot{\mathbf{Q}} + 2\Omega\dot{\mathbf{Q}} + \Omega^2\mathbf{Q} + \dot{\Omega}\mathbf{Q}). \quad (5.6)$$

There are three “extra” terms here. The third term is due to the angular acceleration of the rotating frame.

**PROPOSITION 5.8.** *If the rotating frame  $\mathcal{F}'$  is rotating uniformly, then the absolute and apparent accelerations are related by*

$$\mathbf{A} = R^T \mathbf{a} - 2\boldsymbol{\omega} \times \mathbf{V} + \boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega}, \quad (5.7)$$

where  $\mathbf{A}$  is the apparent acceleration and  $\mathbf{a}$  the absolute acceleration.

The first term  $R^T \mathbf{a} = R^{-1} \mathbf{a}$  is the actual (absolute) acceleration expressed in terms of the moving frame. The two extra terms are new components and are given names:

- $-2\boldsymbol{\omega} \times \mathbf{V} = -2\boldsymbol{\Omega} \mathbf{V}$  is the **Coriolis** acceleration
- $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{Q}) = \boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega}$  is the **centrifugal** acceleration.

**FICTITIOUS FORCES** If we think in terms of Newton's law and forces, suppose the particle has mass  $m$  and is subject to a force  $\mathbf{F}$  as measured in the inertial frame. Then  $\mathbf{F} = m\mathbf{a}$ . In the rotating frame instead, we would like to have  $\mathbf{F}' = m\mathbf{A}$ , so that (with  $R = I$ )

$$\mathbf{F}' = \mathbf{F} + 2m\mathbf{V} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega}. \quad (5.8)$$

and now  $2m\mathbf{V} \times \boldsymbol{\omega}$  is called the **Coriolis force**, and  $m\boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega}$  the **centrifugal force**. The Coriolis and centrifugal forces are often called **fictitious forces** because the bodies don't actually feel them, they are due only to the observer's motion.

The Coriolis force  $2m\mathbf{V} \times \boldsymbol{\omega}$  depends on the velocity and is perpendicular to it. On the Earth, it is the main factor determining how winds blow—the motion of the air is caused by the pressure gradient, but you may have noticed that on weather maps the wind moves *along* the isobars (curves of constant pressure) and not across them; this is due to the Coriolis effect. It is left to the reader to determine from (5.8) whether the Coriolis force deflects the wind to the left or to the right in the Northern hemisphere: it is the opposite in the Southern hemisphere.

The centrifugal force is the “force” you feel pushing you outwards when turning a corner in a car. Of course, there is no such force, it is just your inertia trying to maintain straight line motion (your desire to adhere to Galileo's law of inertia).

**EXAMPLE 5.9.** Consider 2 bodies of mass  $m_1$  and  $m_2$  interacting by gravitational attraction. The force is given by Newton's law of gravitation:

$$F = \frac{Gm_1m_2}{R^2}$$

where  $R$  is the distance between the two bodies. Consider a rotating frame  $\mathcal{F}'$  based at the centre of mass and rotating with angular speed  $\omega$ , and suppose that in this frame the two bodies are at rest. Find  $\omega$  (it will depend on  $R$ ).

*Solution:* The acceleration of body 1 in the rotating frame is given by equation

(5.7) (putting  $R = I$  to line up the axes):

$$\mathbf{A} = \mathbf{a} + 2\mathbf{V} \times \boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega}$$

Now, in the rotating frame the body is required to be at rest, so  $\mathbf{V} = 0 = \mathbf{A}$ . Thus

$$\mathbf{a} + \boldsymbol{\omega} \times \mathbf{q} \times \boldsymbol{\omega} = 0.$$

Using Newton's law,  $\mathbf{F} = m_1 \mathbf{a}$  gives

$$\frac{Gm_2}{R^2} = \omega^2 r_1.$$

Since the origin is the centre of mass, we know that  $m_1 r_1 = m_2 r_2$ . Adding  $m_2 r_1$  to both sides gives  $(m_1 + m_2) r_1 = m_2 (r_1 + r_2)$  so that  $M r_1 = R m_2$ , where  $M = m_1 + m_2$  is the total mass. Finally we deduce

$$\omega^2 = \frac{GM}{R^3}.$$

Note: in the case of the sun and the planets,  $M \approx m_1$  (the mass of the sun,  $M_\odot$  is so very much larger than the mass of a planet), so the expression for  $\omega^2$  above implies Kepler's law that the square of the period of rotation of a planet around the sun is proportional to the cube of its distance from the sun (the period is  $2\pi\omega^{-1}$ ):

$$T^2 = \frac{4\pi^2}{\omega^2} \approx \frac{2\pi^2}{GM_\odot} R^3.$$

[In fact Kepler's law also applies to elliptic motion, which is not covered by our argument.]

In this example, we have found a motion which in the rotating frame is an equilibrium. Such a motion is known as a *relative equilibrium*<sup>a</sup>.

<sup>a</sup>the term *relative equilibrium* was introduced by Henri Poincaré in the early 1900s; sometimes relative equilibria are known as *steady motions*, or even *stationary motions* (!) —see also Section 2.6

### 5.3 LAGRANGIANS IN A ROTATING FRAME

It is useful to be able to write the Lagrangian that describes the motion of a particle or a system in a non-inertial frame. Here we deal just with rotating frames.

So suppose  $\mathcal{F}$  is an inertial frame, and  $\mathcal{F}'$  a frame that is rotating uniformly with respect to  $\mathcal{F}$ , and with common origin, which is the centre of rotation. Let  $\boldsymbol{\omega}$  be the angular velocity of rotation of the frame  $\mathcal{F}'$ .

Starting with the standard Lagrangian for a particle,

$$\mathcal{L}(\mathbf{q}, \mathbf{v}) = \frac{1}{2} m |\mathbf{v}|^2 - V(\mathbf{q}).$$

Recall from §5.2B (see p. 88) that the velocity  $\mathbf{v}$  in the inertial frame is related to the velocity  $\mathbf{V}$  in the rotating frame by

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{q}.$$

(Again, taking  $R = I$ ). Now substitute this into the Lagrangian to get

$$\begin{aligned} \mathcal{L}'(\mathbf{Q}, \mathbf{V}) &= \frac{1}{2} m |\mathbf{V} + \boldsymbol{\omega} \times \mathbf{Q}|^2 - V(\mathbf{Q}) \\ &= \frac{1}{2} m |\mathbf{V}|^2 + m \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{Q} + \frac{1}{2} m |\boldsymbol{\omega} \times \mathbf{Q}|^2 - V(\mathbf{Q}). \end{aligned}$$

**PROPOSITION 5.10.** *In a uniformly rotating frame  $\mathcal{F}'$ , the Lagrangian is written*

$$\mathcal{L}(\mathbf{Q}, \mathbf{V}) = \frac{1}{2} m |\mathbf{V}|^2 + m \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{Q} + \frac{1}{2} m |\boldsymbol{\omega} \times \mathbf{Q}|^2 - V(\mathbf{Q}), \quad (5.9)$$

where  $\mathbf{Q}$  and  $\mathbf{V}$  are the position and velocity as measured in the rotating frame, and  $\boldsymbol{\omega}$  is the angular velocity vector of the rotating frame. With this expression, Lagrange's equations are unchanged.

**PROOF:** We need to check that Lagrange's equations give the correct equations of motion, and for that we need to refer to §5.2B (see p. 88).

First calculate the derivatives that appear in Lagrange's equation: We have

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{V}} \right) = m \frac{d}{dt} (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{Q}) = m \mathbf{A} + \boldsymbol{\omega} \times \mathbf{V}$$

And for the RHS, we need the fact that

$$\frac{\partial}{\partial \mathbf{Q}} |\boldsymbol{\omega} \times \mathbf{Q}|^2 = 2 \boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega}.$$

Then,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{Q}} = m \mathbf{V} \times \boldsymbol{\omega} + m \boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega} - \vec{\nabla} V.$$

So Lagrange's equation becomes

$$m \mathbf{A} + \boldsymbol{\omega} \times \mathbf{V} = m \mathbf{V} \times \boldsymbol{\omega} + m \boldsymbol{\omega} \times \mathbf{Q} \times \boldsymbol{\omega} - \vec{\nabla} V,$$

which with  $-\vec{\nabla} V = \mathbf{F}$  simplifies to give the expression in §5.2C (equation (5.6)).  $\square$

**EXAMPLE 5.11. (Bead on rotating hoop)**

Consider a bead constrained to lie on a smooth circular hoop, that itself is rotating at a constant angular speed of  $\omega$  rad/sec. Let  $m$  be the mass of the bead, and  $r$  the radius of the hoop. This is a one-degree of freedom problem, and we use  $\theta$  which measures the angle between the position vector of the bead and the downward vertical as the configuration coordinate.

In the frame rotating with the hoop, at angular speed  $\omega$ , the kinetic energy of the bead is

$$T = \frac{1}{2} m \ell^2 \dot{\theta}^2,$$

and the potential energy is

$$V = -mgr \cos \theta.$$

The position vector of the bead is

$$\mathbf{r} = (r \sin \theta, 0, -r \cos \theta)^T$$

so  $\mathbf{v} = \dot{\mathbf{r}} = (r \cos \theta, 0, -r \sin \theta)^T \dot{\theta}$ . Then  $\mathbf{v} \cdot \boldsymbol{\omega} \times \mathbf{r} = 0$  and from equation (5.9) the Lagrangian is therefore

$$\mathcal{L} = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \omega^2 r^2 \sin^2 \theta + mgr \cos \theta. \quad (5.10)$$

Note that  $\boldsymbol{\omega} = (0, 0, \omega)^T$  in both frames.

There are two ways to proceed with the analysis of this problem: firstly we can derive Lagrange's equations of motion, and secondly we can interpret the Lagrangian as one deriving from a potential well problem.

So, first Lagrange's equations follow from:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{d}{dt} (m r^2 \dot{\theta}) = m r^2 \ddot{\theta}$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = m r^2 \omega^2 \sin \theta \cos \theta - mgr \sin \theta.$$

Thus Lagrange's equation is, after dividing by  $m r^2$ ,

$$\ddot{\theta} = \frac{1}{2} \omega^2 \sin(2\theta) - \frac{g}{r} \sin \theta. \quad (5.11)$$

When  $\omega = 0$  this reduces to the equation of motion of the plane pendulum, and it is therefore not surprising that this equation for the bead in a rotating hoop is not solvable analytically. However, one can observe that  $\theta = 0$  and  $\theta = \pi$  are equilibria.

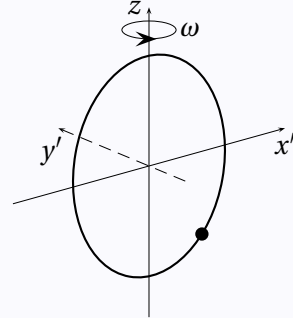


Figure 5.2: Bead on rotating hoop

We proceed now to interpret the Lagrangian  $\mathcal{L}$  in (5.10) above as “kinetic - potential”, with kinetic energy  $T = \frac{1}{2}mr^2\dot{\theta}^2$  and what is called the **effective potential**

$$V_{\text{eff}}(\theta) = -\frac{1}{2}m\omega^2 r^2 \sin^2 \theta - mgr \cos \theta. \quad (5.12)$$

This is not in fact the potential energy of a force field, but appears in the Lagrangian as if it were—hence the name “effective” potential. Remember that  $\omega$  is a constant.

*Dynamics and bifurcations:* First note that the point  $\theta = 0$  is always an equilibrium solution (either from (5.11) because at that point  $\ddot{\theta} = 0$ , or from (5.12) because  $\theta = 0$  is a critical point of  $V_{\text{eff}}$ ). Intuitively, it seems reasonable to expect that if the hoop is not rotating at all, or is rotating slowly then this position would be a stable equilibrium, whereas if the hoop is rotating sufficiently fast, then it would become unstable. Let us check whether this is the case, and if so, at what value of  $\omega$  the transition occurs (the “bifurcation”).

There are basically 2 (related) ways of determining whether an equilibrium point is stable. Firstly by studying the linearized equations, and seeing if all the solutions are oscillatory, and secondly by studying the type of critical point of the (effective) potential energy. If it has a local minimum then the equilibrium is stable. We consider both of these methods in turn, arriving at the same conclusion each time.

The linearization of the equation of motion (5.11) at  $\theta = 0$  is

$$\ddot{\theta} = -\left(\frac{g}{r} - \omega^2\right)\theta.$$

Thus, when

$$\omega^2 < \frac{g}{r}$$

the equilibrium position  $\theta = 0$  is stable, and the small oscillations occur with period approximately

$$T = \frac{2\pi}{\sqrt{(g/r) - \omega^2}}.$$

On the other hand, when  $\omega^2 > g/r$ , the equilibrium point at  $\theta = 0$  is unstable. So what happens to the trajectories? Starting at  $\theta \approx 0$ , the bead will move away from the unstable equilibrium, and will oscillate about another equilibrium. To find this other equilibrium, use the differential equation above (5.11) for  $\ddot{\theta} = \dot{\theta} = 0$ . We need to solve

$$\frac{1}{2}\omega^2 \sin(2\theta) - \frac{g}{r} \sin \theta = 0.$$

But with  $\sin \theta \neq 0$  this becomes (expanding  $\sin(2\theta)$  and dividing by  $\sin \theta$ ):

$$\omega^2 \cos \theta - \frac{g}{r} = 0.$$

This equation has solutions provided  $g/(r\omega^2) \leq 1$ , or in other words

$$\omega > \omega_0 \equiv \sqrt{g/r}.$$

The solutions are of course

$$\theta = \theta_e := \pm \cos^{-1}\left(\frac{g}{r\omega^2}\right).$$

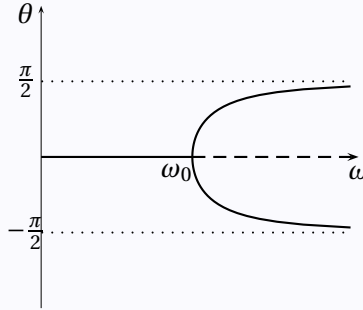


Figure 5.3: Bifurcation diagram for a bead on a rotating hoop. Solid lines represent stable equilibria, broken lines unstable ones.

Linearizing at this point gives, where we write  $\phi = \theta - \theta_e$

$$\ddot{\phi} = \left(\omega^2 \cos(2\theta_e) - \frac{g}{r} \cos \theta_e\right) \phi$$

But  $\cos(\theta_e) = g/(r\omega^2)$  and  $\cos(2\theta_e) = 2\cos^2\theta_e - 1 = 2g^2/(r^2\omega^4) - 1$ , so

$$\ddot{\phi} = -\frac{r^2\omega^4 - g^2}{r^2\omega^2} \phi$$

which describes oscillations with period

$$T = \frac{2\pi r\omega}{\sqrt{r^2\omega^4 - g^2}}.$$

This information is summarized in a bifurcation diagram in Figure 5.11.

Alternatively, the stability or instability of the equilibrium points can also be established using the potential energy. Equilibria occur where  $\vec{\nabla} V = 0$ , which here is just  $\frac{dV}{d\theta} = 0$ , which translates to

$$\frac{dV}{d\theta} = -m\omega^2 r^2 \sin\theta \cos\theta + mgr \sin\theta = 0.$$

This has solutions when  $\sin\theta = 0$  and  $\cos\theta = g/(\omega^2 r)$ , which is precisely what was found above. Furthermore, the stability of each equilibrium is determined by whether  $V$  has a local minimum or a local maximum at  $\theta_e$ . Now,

$$\frac{d^2V}{d\theta^2} = -m\omega^2 r^2 (\cos^2\theta - \sin^2\theta) + mgr \cos\theta.$$

For  $\theta = 0$ , we see

$$V''(0) = -m\omega^2 r^2 + mgr$$

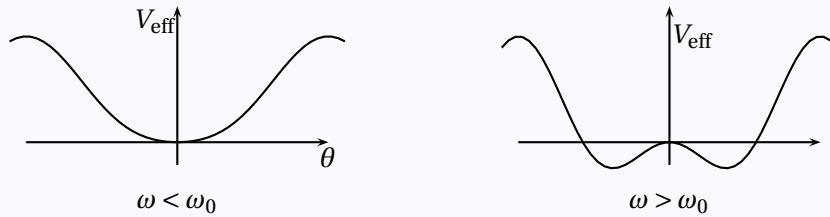
which is positive provided  $\omega < \sqrt{g/r} \equiv \omega_0$ .

When  $\omega > \omega_0$ , the other equilibria occur at  $\cos\theta_e = g/(\omega^2 r)$ , and satisfy

$$\begin{aligned} V''(\theta_e) &= -m\omega^2 r^2 (\cos^2 \theta_e - \sin^2 \theta_e) + mgr \cos \theta_e \\ &= -m\omega^2 r^2 (2(g^2)/(\omega^4 r^2) - 1) + mg^2/\omega^2 \\ &= \frac{m}{\omega^2} (\omega^4 r^2 - g^2) > 0. \end{aligned}$$

These other equilibria at  $\theta = \theta_e$  are therefore always stable, whenever they exist (ie, whenever the hoop is rotating sufficiently fast:  $\omega^2 > g/r$ ). This agrees with what was found above using the linearization. However this method does not tell us the period of small oscillations, because this depends on more than just the (effective) potential energy function.

To visualize this better, let us plot the effective potential as a function of  $\theta$ :



Notice how when  $\omega < \omega_0$  there is just one minimum, whereas if  $\omega > \omega_0$  there are two, and a maximum between them at  $\theta = 0$ .

**Remark** The Lagrangian for this example can be derived without using rotating frames, but simply from the usual kinetic minus potential formulation, though the kinetic energy must of course take into account the fact that the hoop is rotating.

## PROBLEMS

- 5.1 (a) Give an example to show that in general  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ , but show that for any two vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ ,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \mathbf{a} \times (\mathbf{b} \times \mathbf{a})$$

This allows us to write  $\mathbf{a} \times \mathbf{b} \times \mathbf{a}$  with no ambiguity.

[Hint: use  $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ ]

(b) Show that  $\boldsymbol{\omega} \times \mathbf{r} \times \boldsymbol{\omega}$  is a scalar multiple of the projection of  $\mathbf{r}$  onto the plane perpendicular to  $\boldsymbol{\omega}$  (the scalar is  $\|\boldsymbol{\omega}\|^2$ ).

- 5.2 Let  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ . Show that  $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$ , where  $\mathbf{u} \cdot \mathbf{v}$  is the scalar (dot) product, and  $\mathbf{u}^T \mathbf{v}$  is the product (in the sense of matrix multiplication) of the row vector  $\mathbf{u}^T$  and the

column vector  $\mathbf{v}$ . Deduce that if  $A$  is an  $n \times n$  matrix, then,

$$\mathbf{u} \cdot (A\mathbf{v}) = \mathbf{u}^T A\mathbf{v} = \mathbf{v}^T A^T \mathbf{u}.$$

In the following three questions, let  $\Omega$  be the skew-symmetric matrix,

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

**5.3** Let  $\boldsymbol{\omega} = (a, b, c)$ . Show that (as stated in Lemma 5.7)

$$\Omega \mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}, \quad \forall \mathbf{u} \in \mathbb{R}^3.$$

Show directly that  $\Omega \boldsymbol{\omega} = 0$ . Show also that if  $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$  then there is a vector  $\mathbf{u} \in \mathbb{R}^3$  for which  $\boldsymbol{\omega}_1 \times \mathbf{u} \neq \boldsymbol{\omega}_2 \times \mathbf{u}$ , and deduce that  $\boldsymbol{\omega}$  is uniquely determined by the equation above.

**5.4** Find the characteristic polynomial  $\chi(\lambda)$  of  $\Omega$ , and deduce that the eigenvalues are 0 and  $\pm i\sqrt{a^2 + b^2 + c^2}$ . Use Exercise 5.3 to find the eigenvector for the eigenvalue 0. How are the non-zero eigenvalues related to the vector  $\vec{\omega}$  associated to  $\Omega$ ?

**5.5** Show that if  $A$  is any rotation matrix, then under the correspondence  $\Omega \mapsto \boldsymbol{\omega}$ , one has that  $A\Omega A^{-1}$  is taken to  $A\boldsymbol{\omega}$ . [Hint: You may use the fact that if  $A$  is a rotation matrix then  $A(\mathbf{a} \times \mathbf{b}) = (A\mathbf{a}) \times (A\mathbf{b})$ . To see this, think about the geometric definition of the vector product.]

**5.6** If a wind is blowing in the Northern hemisphere on Earth, the Coriolis force would deflect it: in which direction would it be deflected, right or left?

**5.7** A person sitting at the edge of a rotating disc fires a gun aiming directly towards the centre of the disc. Will the bullet pass through the centre? Draw a diagram to illustrate your answer.

To be specific, suppose the disc is rotating anticlockwise with angular velocity 0.1 rad/sec, and the gun is at a distance 100m from the centre (a big disc!) and the bullet is fired at a velocity of 100m/s (a slow bullet). What is the velocity of the bullet as observed in an inertial frame?

Find the Coriolis and centrifugal accelerations as observed in the rotating frame at the moment of firing the bullet.

**5.8** Let  $A(t)$  be a matrix function of  $t$  so that  $A(t)$  is always invertible. Let  $B(t) = A(t)^{-1}$ . By differentiating the relation  $B(t)A(t) = I$  show that

$$\dot{B} = -A^{-1} \dot{A} A^{-1}.$$

How is this related to  $\frac{d}{dt} \left( \frac{1}{a(t)} \right)$  for  $a(t)$  a scalar function?