CHAPTER 2

LAGRANGE'S EQUATIONS

THERE ARE BASICALLY TWO approaches to conservative mechanics, due principally to Lagrange and Hamilton respectively. Each has advantages over the other, although the Lagrangian approach is almost always important as a first step in finding the Hamiltonian framework. Newton's description of mechanics really requires one to be working in Euclidean (Cartesian) coordinates, and Lagrange[1] found a very elegant approach which allows one to use different coordinate systems. In this chapter we describe this approach; this approach has the added advantage of making simple constraints easy to handle.

2.1 LAGRANGE'S EQUATIONS OF MOTION

To study the motion of a system of N particles described in Chapter 1, one starts with Newton's second law which states:

$$\ddot{\mathbf{r}}_i = \frac{1}{m_i}\mathbf{F}_i^{\text{tot}}, \tag{2.1}$$

where $\mathbf{F}_i^{\text{tot}}$ is the total force acting on particle i ($i = 1,\ldots,N$), and $\mathbf{r}_i$ is the position vector of particle i. This is a system of $3N$ second order differential equations in $3N$ variables $(x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$. The space of all possible configurations is therefore $\mathbb{R}^{3N}$ and is called the ***configuration space*** of the system. One could alternatively be considering N particles moving in the plane $\mathbb{R}^2$, and the configuration space would then be $\mathbb{R}^{2N}$ or N particles moving on a line, in which case it would be $\mathbb{R}^N$. Recall that the total number of coordinates required is called the number of ***degrees of freedom*** of the system. In this book, we deal only with conservative forces, so that we assume there is a function on configuration space $V(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ called the potential (energy), such that $\mathbf{F}_i^{\text{tot}} = -\nabla_i V$, where as usual $\nabla_i V = \partial V/\partial \mathbf{r}_i = (\partial V/\partial x_i,\, \partial V/\partial y_i,\, \partial V/\partial z_i)^T$.

[1] Joseph-Louis Comte de Lagrange lived 1737–1813, and published the first edition of his great treatise *Mécanique Analytique* (Analytical Mechanics) in 1788, just 100 years after Newton's *Principia* (1687)

EUCLIDEAN COORDINATES

Although one of the main advantages of Lagrange's approach over Newton's is the flexibility in the coordinate system used, we begin by introducing Lagrange's approach in Euclidean coordinates.

Lagrange found he could formulate Newton's laws in a manner that did not depend on using Euclidean coordinates, and he was led to introduce a function equal to the *difference* between the kinetic and potential energies, now called the ***Lagrangian***. In Euclidean coordinates this is,

$$\boxed{\mathcal{L}(q,\mathbf{v}) = T(\mathbf{v}) - V(q)} \tag{2.2}$$

where

$$\begin{aligned} q &= (\mathbf{r}_1,\ldots,\mathbf{r}_N) \in \mathbb{R}^{3N} \\ \mathbf{v} &= (\mathbf{v}_1,\ldots,\mathbf{v}_N) \in \mathbb{R}^{3N} \\ T(\mathbf{v}) &= \tfrac{1}{2}\sum_i m_i \|\mathbf{v}_i\|^2 \end{aligned}$$

and V is the potential energy. The set $\{(q,\mathbf{v})\}$ of all possible configurations[2] and velocities is called the ***Lagrangian phase space*** and the Lagrangian is therefore a function on this phase space

$$\mathcal{L} : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R}.$$

DEFINITION 2.1. Given any smooth function $\mathcal{L}(q,\mathbf{v})$ on the phase space, ***Lagrange's equations*** are

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right) = \frac{\partial \mathcal{L}}{\partial q}} \tag{2.3}$$

This is a vector expression with $3N$ components, so consists of $3N$ differential equations, one for each component of $\mathbf{q}$, each of second order in $\mathbf{q}$, so

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial \mathcal{L}}{\partial v_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \quad i = 1,2,\ldots,3N$$

(or $i = 1,\ldots,2N$ if the motion is in a plane, etc). ✔

In practice, when solving a problem, one writes the $3N$ equations out one by one.

Since $\mathbf{v} = \dot{\mathbf{q}}$, Lagrange's equations are often written

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) = \frac{\partial \mathcal{L}}{\partial q}} \tag{2.4}$$

or one by one,

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \quad i = 1,2,\ldots,3N$$

[2] Note that the symbol q for configuration is not in bold face — we do not think of configurations as vectors, but just as coordinates. Addition of configurations does not (usually) have any meaning. On the other hand velocity $\dot{\mathbf{q}}$ *is* a vector

Remark 2.2. Lagrange's equations are also known as the ***Euler-Lagrange equations***. This is because there is an important interpretation of Lagrange's equations as a variational problem, the so-called "principle of least action", see Section 2.7. ❞

Example 2.3. (Lagrangian for a simple spring) Consider a particle of mass m on the end of a spring of spring constant k, and let $x \in \mathbb{R}$ be the extension of the spring. Since the configuration of the system is determined by a single variable x, it is a system with 1 degree of freedom. The Lagrangian phase space is just $\mathbb{R} \times \mathbb{R}$ in this example, so $q = x$ and $\mathbf{v} = v = \dot{x}$. Then

$$T(v) = \tfrac{1}{2} m v^2, \quad V(x) = \tfrac{1}{2} k x^2.$$

Consequently, the Lagrangian is simply

$$\mathcal{L}(x, v) = T - V = \tfrac{1}{2} m v^2 - \tfrac{1}{2} k x^2,$$

and Lagrange's equation requires:

$$\frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{\partial \mathcal{L}}{\partial v}\right) = \frac{\mathrm{d}}{\mathrm{d}\, t}(m v) = m \dot{v} = m \ddot{x},$$

and

$$\frac{\partial \mathcal{L}}{\partial x} = -k x$$

so that Lagrange's equation gives us

$$m \ddot{x} = -k x,$$

which is (of course!) the same as the equation one obtains directly from Newton's second law (as $F = -kx$ for a spring).

This example is indicative of the general equivalence:

Proposition 2.4. *In Cartesian coordinates, Lagrange's equations (2.3) are equivalent to Newton's equations (2.1).*

Proof: This is an easy calculation: for $\partial \mathcal{L}/\partial \mathbf{v}_i = m_i \mathbf{v}_i = \mathbf{p}_i$, and $\partial \mathcal{L}/\partial \mathbf{r}_i = -\operatorname{grad}_i V$. Lagrange's equation is then just $\dot{\mathbf{p}}_i = -\operatorname{grad}_i V$ as required. ❐

Remark 2.5. Notice that $\partial \mathcal{L}/\partial \mathbf{v} = \mathbf{p}$, the momentum, and $\partial \mathcal{L}/\partial q = -\operatorname{grad} V$ which is the force. We will return to this later. ❞

Example 2.6. Find the equations of motion of a particle of mass m moving in the plane with potential energy $V(x, y) = \frac{1}{2}(x^2 + y^2)$.

Solution: We need to find the Lagrangian. The kinetic energy is $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$, so that

$$\mathcal{L}(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}(x^2+y^2).$$

Lagrange's equations give us:

- for x:
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\mathrm{d}(m\dot{x})}{\mathrm{d}\,t} = m\ddot{x},$$
and
$$\frac{\partial\mathcal{L}}{\partial x} = -x.$$
Lagrange's equation for x is then $m\ddot{x} = -x$.
- for y: this is analogous —
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = \frac{\mathrm{d}(m\dot{y})}{\mathrm{d}\,t} = m\ddot{y}, \qquad \frac{\partial\mathcal{L}}{\partial y} = -y.$$

The equations of motion are therefore

$$m\ddot{x} = -x, \qquad m\ddot{y} = -y. \tag{2.5}$$

EXAMPLE 2.7. (Two body problem) Consider 2 particles of masses m_1, m_2, moving in space under a mutual gravitational interaction (a system with 6 degrees of freedom). Let $\mathbf{r}_1, \mathbf{r}_2$ be the position vectors of the two particles. The gravitational potential energy is determined by their mutual distance:

$$V(\mathbf{r}_1,\mathbf{r}_2) = -\frac{Gm_1m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|},$$

where G is Newton's "universal gravitational constant". The kinetic energy is

$$T(\mathbf{v}_1,\mathbf{v}_2) = \tfrac{1}{2}m_1\|\mathbf{v}_1\|^2 + \tfrac{1}{2}m_2\|\mathbf{v}_2\|^2,$$

so that the Lagrangian is given by

$$\mathcal{L}(\mathbf{q},\mathbf{v}) = \tfrac{1}{2}m_1\|\mathbf{v}_1\|^2 + \tfrac{1}{2}m_2\|\mathbf{v}_2\|^2 + \frac{Gm_1m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|}. \tag{2.6}$$

Then for $j = 1,2$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}_j}\right) &= m_j\dot{\mathbf{v}}_j = \dot{\mathbf{p}}_j \\ \frac{\partial\mathcal{L}}{\partial\mathbf{r}_j} &= -\frac{Gm_1m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|^3}(\mathbf{r}_j-\mathbf{r}_k),\end{aligned}$$

where $k \neq j$. Lagrange's equations are therefore

$$\begin{cases} \dot{\mathbf{p}}_1 &= \frac{Gm_1m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|^3}(\mathbf{r}_1-\mathbf{r}_2) \\ \dot{\mathbf{p}}_2 &= \frac{Gm_1m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|^3}(\mathbf{r}_2-\mathbf{r}_1) \end{cases}$$

which are of course the same as Newton's equations.

2.2 More general coordinates

It may seem that we have gained little in rewriting Newton's equation in Lagrange's form. However the power of Lagrange's insight is in the following. Suppose we no longer use Euclidean (Cartesian) coordinates, but some others such as polar coordinates, or spherical polar coordinates (which might describe particles moving on the surface of a sphere)—classically such non-Euclidean coordinates were called ***generalized coordinates***—then Newton's laws no longer hold, at least not in the form $m\ddot{\mathbf{q}} = -\operatorname{grad} V$. Lagrange wanted to write the equations of motion in a manner that did not depend upon using Euclidean coordinates, and we now proceed to show how Lagrange's equations do this. There is another important advantage to Lagrange's equations, which is that it can deal with many constrained systems, as we will see in the next section.

First we go through the argument with an example, and then proceed to adapt the example to deal with the general setting.

Example 2.8. Let us redo example 2.6 but in polar coordinates. We have $V(x,y) = \frac{1}{2}(x^2+y^2)$, so in polar coordinates, $V(r,\theta) = \frac{1}{2}r^2$ (it happens to be independent of θ). The kinetic energy is (see Problem 1.8), $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$. Thus,

$$\mathcal{L}(r,\theta,\dot{r},\dot{\theta}) = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \tfrac{1}{2}r^2.$$

Lagrange's equations give us:

- for r:
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{\mathrm{d}(m\dot{r})}{\mathrm{d}\,t} = m\ddot{r},$$
and (don't forget the r term in the kinetic energy)
$$\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - r.$$
Thus $m\ddot{r} = r\dot{\theta}^2 - r$.
- for θ:
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\mathrm{d}}{\mathrm{d}\,t}(mr^2\dot{\theta}) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta},$$

(using the product rule). On the other hand $\frac{\partial \mathcal{L}}{\partial \theta} = 0$, giving $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0$.

Thus in polar coordinates the equations of motion are,

$$\begin{cases} m\ddot{r} &= mr\dot{\theta}^2 - r \\ r\ddot{\theta} &= -2\dot{r}\dot{\theta}. \end{cases} \tag{2.7}$$

The question is, are these the equations of motion arising from Newton's laws (as in Example 2.6)? The answer is, of course, affermative. See Problem 2.2.

Recall that kinetic energy is defined using Cartesian coordinates, and for this reason in all cases, we must start with Cartesian coordinates. Let us write $\mathbf{x} = (x_1, x_2, \ldots, x_{3N})$ to describe the Cartesian coordinates of all the N particles in the system.

Next write $q = (q_1, \ldots, q_n)$ for the new coordinates (polar, spherical, whatever,...)—for a system with n degrees of freedom. Then write ψ for the transformation converting non-Cartesian coordinates to Cartesian ones:

$$\mathbf{x} = \psi(q).$$

Thus ψ is a smooth[3] map $\psi : \mathbb{R}^n \to \mathbb{R}^{3N}$.

For example, using polar coordinates for a single particle moving in the plane, we have $x = r\cos\theta$ and $y = r\sin\theta$ and $z = 0$, so that

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \psi(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}. \tag{2.8}$$

Here $3N = 3$ and $n = 2$.

To use Lagrange's equations in the new coordinates q requires us to derive the expression for kinetic energy in the new coordinates, for example using $\dot{r}$, $\dot{\theta}$ in polar coordinates (as in Problem 1.8). Now using the coordinates q to describe the motion means that we will use

$$\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)^T$$

as velocity vector. And we want to express the kinetic energy in terms of $\dot{\mathbf{q}}$ rather than the $\mathbf{v} = \dot{\mathbf{x}}$ used in its definition.

If $\mathbf{x} = \psi(q)$ and q depends on time t, then $\mathbf{x}(t) = \psi(q(t))$ and the velocity is related by differentiating with respect to t:

$$\dot{\mathbf{x}} = D\psi(q)\,\dot{\mathbf{q}}, \tag{2.9}$$

by the chain rule, where $D\psi(q)$ is the ***Jacobian matrix*** of ψ at q. It is the $3N \times n$ matrix, defined by

$$(D\psi)_{ij} = \frac{\partial \psi_i}{\partial q_j}.$$

[3] smooth means ψ can be differentiated as many times as we want; in practice 2 or 3 times differentiable is sufficient

(That's the ith row and jth column.) For example, for polar coordinates we have from (2.8) that

$$D\psi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 0 & 0 \end{pmatrix}. \tag{2.10}$$

The relation between $\dot{\mathbf{q}}$ and $\dot{\mathbf{x}}$ is then

$$\dot{\mathbf{x}} = D\psi(q)\,\dot{\mathbf{q}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ 0 \end{pmatrix};$$

that is, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$ (and $\dot{z} = 0$), as we saw above.

Remark 2.9. Notice that q is not considered to be a vector, although it does have several components—it is meaningless to add polar coordinates, or multiply them by a scalar. On the other hand, the velocity $\dot{\mathbf{q}}$ is a vector: addition and scalar multiplication of velocities *is* meaningful. That said, we do write $\mathbf{x}$ (Cartesian coordinates) as a vector. ❞

The next step is to write the kinetic energy in terms of $\dot{\mathbf{q}}$, similar to that for polar coordinates in the example above. However, since the approach is the same for constrained systems, we postpone the general case until after describing constrained systems.

2.3 Constrained systems

Many systems involve constraints of some kind. If there are no constraints, then each particle is free to move in 3-dimensions, so with N particles there are $3N$ degrees of freedom. Constraints decrease the number of degrees of freedom. Here we list some of the principal type of constraint:

1. A particle on a plane, for example a smooth table. Here the constraint is "lying on the table", and in Cartesian coordinates imposes the condition $z = 0$, so reducing the number of degrees of freedom from 3 to 2. The system could be parametrized by (x, y), or using polar coordinates, by (r, θ).

2. In a planar pendulum, firstly the bob is constrained to move in a vertical plane, so reducing the number of degrees of freedom from 3 to 2, but then the inelastic rod imposes another (independent) condition that the distance of the bob from the fixed point of the rod is constant, equal to ℓ, in other words one has $x^2 + y^2 = \ell^2$ (in appropriate coordinates). The end result is that it is sensible to use just the angle θ as configuration variable, and there is just the one degree of freedom.

3. A bead moving on a smooth wire: this has just 1 degree of freedom, and the bead is constrained to the wire by appropriate equations. The system could be parametrized by the variable parametrizing the curve.

4. A cylinder rolling down an inclined plane.

5. A ball rolling on a table top.

The constraint in each of the last two examples involves velocity: the constraint of ***rolling*** is expressed by saying that the point of contact at any instant has zero velocity. This type of constraint is harder to deal with than the constraints which only involve position in their definition, like those numbered 1,2,3 in the list above.

These latter constraints which arise as restrictions on the possible *configurations* and are called ***holonomic constraints***. The harder ones involving the velocities as well as the configurations are therefore called non-holonomic constraints. We will just be dealing with holonomic constraints.

A simple example of a constrained system is the plane pendulum as described above: while the bob moves in the plane, it is constrained by the inextensible rod to lie on a circle. Similarly in the Atwood machine (see Problem sheet 1), the two masses are constrained to lie at a fixed distance apart (distance as measured up over the pulley and down again—equal to the length of the inextensible string). In fact almost all the examples we are interested in have some constraints, and in order to proceed further we must therefore include these constraints in the Lagrangian formalism.

As in the previous section, we will choose coordinates adapted to the problem, and which moreover respect the constraints. For example, for the plane pendulum we only need the one coordinate θ, and the expression in Cartesian coordinates is then $x = \ell\sin\theta$ and $y = -\ell\cos\theta$ (see the diagram on p. 33).

For a general (constrained) system, we introduce coordinates $q = (q_1, q_2, \ldots, q_n)$. The number n is called the number of ***degrees of freedom*** of the system. So for example, the pendulum with $q = \theta$ has just 1 degree of freedom.

As in the previous section, we will parametrize the (allowed) configurations by a map ψ, so $\psi : \mathbb{R}^n \to \mathbb{R}^{3N}$, with $\mathbf{x} = \psi(q)$. The (allowed) velocities are then given by

$$\mathbf{v} = D\psi(q)\dot{\mathbf{q}}, \tag{2.11}$$

with $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)^T$ (a column vector). Again, $D\psi$ is the Jacobian matrix of ψ, but now it is a $3N \times n$ matrix ($3N$ rows and n columns).

For example, for a simple pendulum the parametrization is $\mathbf{x} = \psi(\theta) = \begin{pmatrix} \ell\sin\theta \\ -\ell\cos\theta \end{pmatrix}$, so the velocity is

$$\mathbf{v} = D\psi(\theta)\dot{\theta} = \begin{pmatrix} \ell\cos\theta \\ \ell\sin\theta \end{pmatrix}\dot{\theta},$$

and geometrically one can see that this velocity is *tangent* to the circle at the point $\psi(\theta)$.

Let us write $Q \subset \mathbb{R}^{3N}$ for the image of ψ; that is Q is the set of configurations that are allowed by the constraints—it is called the ***configuration space***. We are going to assume that Q is a ***regular*** subset[4] of $\mathbb{R}^{3N}$, which happens when the Jacobian matrix $D\psi$ always has rank equal to n (so the linear map $\mathbf{u} \mapsto D\psi\,\mathbf{u}$ is always injective).

The space of ***allowed velocities*** at $\mathbf{x} = \psi(q)$ is the space

$$T_{\mathbf{x}}Q = \{D\psi(q)\,\dot{\mathbf{q}} \mid \dot{\mathbf{q}} \in \mathbb{R}^n\}.$$

Vectors in $T_{\mathbf{x}}Q$ are called tangent vectors to Q at $\mathbf{x}$. (and in geometry, the space $T_{\mathbf{x}}Q$ is called the *tangent space* to Q at $\mathbf{x} = \psi(q)$). The allowed velocities are called "virtual velocities" in some texts.

[4] also called a *submanifold*

In other words, $T_{\mathbf{x}}Q$ is the linear subspace of $\mathbb{R}^{3N}$ equal to the image of the Jacobian matrix $D\psi(q)$, where $\mathbf{x} = \psi(q)$. It is the space of allowed velocities, by equation (2.11)

Definition 2.10. A ***constraint force*** $\mathbf{F}$ on a system is one which is orthogonal to the allowed velocities of the system:

$$\sum_{j=1}^{N} \mathbf{F}_j \cdot \dot{\mathbf{x}}_j = 0.$$

Here the vector $\mathbf{F}_j$ is the effect of the force on particle j. In terms of $\dot{\mathbf{q}}$ this condition is,

$$\mathbf{F} \cdot D\psi_q(\dot{\mathbf{q}}) = 0,$$

for all $\dot{\mathbf{q}} \in \mathbb{R}^n$, where now $\mathbf{F}$ is written as a vector in $\mathbb{R}^{3N}$. Or again this is equivalent to

$$\mathbf{F}^T D\psi(q) = 0,$$

(product of row vector and matrix). ✔

The reson for this definition of constraint force is the following property.

Proposition 2.11. *A constraint force does no work on the system, so in particular does not alter the kinetic energy.*

Proof: The kinetic energy of the system is $T = \sum_{j=1}^{N} \frac{1}{2} m_j \|\mathbf{v}_j\|^2$. Then due to the force $\mathbf{F}$ we have,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \sum_{j=1}^{N} m_j \mathbf{v}_j \cdot \dot{\mathbf{v}}_j = \sum \mathbf{v}_j \cdot \mathbf{F}_j$$

($\mathbf{F}_j = m\dot{\mathbf{v}}_j$ by Newton's second law). So it follows from the definition of constraint force that $\mathrm{d}T/\mathrm{d}t = 0$. ❒

This fact is behind why Lagrange's method works for constrained systems: the constraints do not effect the kinetic energy, and Lagrange's method is based on kinetic energy.

Example 2.12. Let us show directly that the tension in the string in the Atwood machine (see Problem sheet 1) is a constraint force. Now, two masses are moving with the same speed in opposite directions, write $\mathbf{v}_m = -\mathbf{v}_M$. Both are vertical vectors so write $\mathbf{v}_M = v\mathbf{e}_3$. The tension in the string is the same for each of the masses, say $T\mathbf{e}_3$ ($T > 0$ as both act upwards). Then

$$(T\mathbf{e}_3) \cdot \mathbf{v}_M + (T\mathbf{e}_3) \cdot \mathbf{v}_m = (T\mathbf{e}_3) \cdot (v - v)\mathbf{e}_3 = 0.$$

Therefore the tension is a constraint force. Note that the tension in the (inextensible) string is what is forcing the speeds of the masses to be equal: if the string were elastic, then the tension would not be a constraint force, and indeed kinetic energy would be converted into elastic energy.

The fact that the velocities are equal and opposite in this example is what is meant by *allowed* velocities in the definition. Similarly in the plane pendulum, any allowed velocity is tangent to the circle of motion (which is perpendicular to the tension in the rod).

More generally, if a system of N particles moves in a way so that all the inter-particle distances are constant (for example they are all connected by light inextensible rods) then the forces maintaining their constant separations (tensions in said rods) are constraint forces.

2.4 LAGRANGIANS IN GENERAL COORDINATES

Whether it is just that the problem suggests using non-Cartesian coordinates, or that the system is constrained is some way forcing a use of other coordinates, in order to describe the state of the system one often needs to introduce some different (non-Cartesian) coordinates. We will denote the new coordinates by $q = (q_1, q_2, \dots, q_n)$. For example, for the plane pendulum one only needs a single variable θ, so in that case $q = \theta$. If we have two particles in the plane, and want to use polar coordinates, then we would have coordinates $q = (r_1, \theta_1, r_2, \theta_2)$.

The Cartesian coordinates $\mathbf{x}$ will depend on the chosen coordinates q through some (differentiable) map $\psi : \mathbb{R}^n \to \mathbb{R}^{3N}$, with $\mathbf{x} = \psi(q)$. The number n is the number of ***degrees of freedom*** of the system.

In these new coordinates, the new Lagrangian is as before equal to "kinetic - potential". The new potential is simply obtained by writing V in terms of q instead of $\mathbf{x}$, and the new kinetic energy requires writing $\mathbf{v}$ (or $\dot{\mathbf{x}}$) in terms of $\dot{\mathbf{q}}$ as in (2.11). In other words we substitute

$$(\mathbf{x}, \dot{\mathbf{x}}) = (\psi(q), D\psi(q)\dot{\mathbf{q}}).$$

That is,

$$\mathcal{L}(q, \dot{\mathbf{q}}) = T(\psi(q), D\psi(q)\dot{\mathbf{q}}) - V(\psi(q)).$$

We call this the Lagrangian on the configuration space.

THEOREM 2.13. *Lagrange's equations on the configuration space are equivalent to Newton's laws with the constraint.*

There are several ways of proving this theorem, but the most beautiful is the one using Hamilton's "principle of least action" and is given in § 2.7.

The basic example of a constraint force is any set of forces on the particles that maintain the distances between them, so that the system behaves as a rigid body.

EXAMPLE 2.14. (**The plane pendulum: Lagrangian approach**)

Traditionally, the mathematical pendulum consists of a particle of mass m—the *bob*—attached to one end of a light inextensible rod of length ℓ, the other end of which is attached to a fixed point. The pendulum is free to swing in a plane. We can describe the position of the pendulum by the angle θ the rod makes with the vertical—Figure 2.1. [Here 'light' means the rod has negligible mass and 'inextensible' means that any elastic effects are ignored]

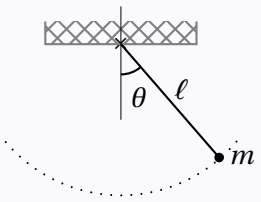


Figure 2.1:
The plane pendulum

As we saw in Chapter 1 (Example 1.11) gravitational force is conservative and the potential depends on the height of the particle above some fixed level. The height above the lowest point of the circle is here $\ell(1-\cos\theta)$, so

$$V = mg\ell(1-\cos\theta).$$

Let us find the Lagrangian in terms of θ and $\dot\theta$. The potential energy is V, and for the kinetic energy we need that the Euclidean coordinates corresponding to θ are

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \ell\sin\theta \\ -\ell\cos\theta \end{pmatrix}.$$

The velocity is then,

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} \ell\cos\theta\,\dot\theta \\ \ell\sin\theta\,\dot\theta \end{pmatrix},$$

so the kinetic energy is

$$T(\dot\theta) = \tfrac{1}{2}m\|\dot{\mathbf{x}}\|^2 = \tfrac{1}{2}m(\dot x^2+\dot y^2) = \tfrac{1}{2}m\ell^2\dot\theta^2.$$

The Lagrangian is therefore

$$\mathcal{L}(\theta,\dot\theta) = \tfrac{1}{2}m\ell^2\dot\theta^2 - mg\ell(1-\cos\theta).$$

Now we write Lagrange's equations (2.3).

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) &= \frac{\partial\mathcal{L}}{\partial\theta} \\ \Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\,t}(m\ell^2\dot\theta) &= -mg\ell\sin\theta \end{aligned}$$

Simplifying (since m and ℓ are constant) gives

$$\ddot\theta = -\frac{g}{\ell}\sin\theta.$$

This is indeed the equation of motion one derives using Newton's law using both the gravitational force and the force due to the tension in the rod (the constraint force: it is always perpendicular to the velocity). Thus, Lagrange's equations continue to hold true even for this constrained system.

Notice that $\partial\mathcal{L}/\partial\dot\theta = m\ell^2\dot\theta$, which is the angular momentum of the pendulum about the origin, and $\partial\mathcal{L}/\partial\theta = -mg\ell\sin\theta$, which is the torque of the gravitational force about the centre of the circle, so in this case Lagrange's equation is saying that torque is equal to rate of change of angular momentum.

KINETIC ENERGY AND THE MASS-INERTIA MATRIX Since the kinetic energy is always quadratic in the velocity vector $\dot{\mathbf{q}}$ (whatever generalized coordinates we are using), it can be written $T(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbb{K}(\mathbf{q})\dot{\mathbf{q}}$, where $\dot{\mathbf{q}}$ is a column vector and $\mathbb{K}(\mathbf{q})$ is a symmetric matrix depending on $\mathbf{q}$, and called the ***mass-inertia matrix*** (also the kinetic matrix).

For example, in polar coordinates the kinetic energy of a particle in the plane is

$$T = \tfrac{1}{2}m(\dot x^2+\dot y^2) = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2).$$

The mass-inertia matrix is then

$$\mathbb{K} = \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix},$$

so that with $\dot{\mathbf{q}} = (\dot r,\, \dot\theta)^T$ one has indeed $\frac{1}{2}\dot{\mathbf{q}}^T\mathbb{K}\,\dot{\mathbf{q}} = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$.

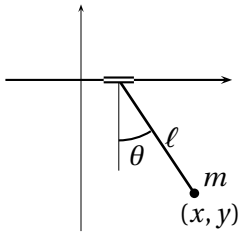


FIGURE 2.2: The elliptic pendulum

EXAMPLE 2.15. (The elliptic pendulum) This consists of a light rod of length ℓ with a bob of mass m at one end, hanging in a vertical plane, with its other end not fixed buts free to slide on a horizontal axis, see Figure 2.2. Determine the Lagrangian and deduce the equations of motion.

Solution: We will use x,θ as coordinates, because specifying the coordinates (x,y) of the bob does not determine the configuration of the pendulum. The relation is $y = -\ell\cos\theta$, so $\dot y = \ell\sin\theta\,\dot\theta$.

First we find the kinetic energy. The only part with mass is the bob, which has Cartesian coordinates (x,y), so the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot x^2+\dot y^2). \\ &= \tfrac{1}{2}m\left(\dot x^2 + \ell^2\sin^2\theta\,\dot\theta^2\right). \end{aligned} \tag{2.12}$$

The mass-inertia matrix is therefore,

$$\mathbb{K}(x,\theta) = \begin{pmatrix} m & 0 \\ 0 & m\ell^2 \sin^2\theta \end{pmatrix}.$$

Notice that when $\sin\theta = 0$ this matrix is degenerate (putting $(\dot{x},\dot{\theta}) = (0,1)$ gives $T = 0$: this corresponds to keeping the mass fixed and just moving the slider infinitesimally, and of course there is no kinetic energy for such a motion). This will have repercussions for the equations of motion.

The potential energy is $V = mgy = -mg\ell\cos\theta$, so the Lagrangian is given by

$$\mathcal{L}(x,\dot{x},\theta,\dot{\theta}) = T - V = \tfrac{1}{2}m\left(\dot{x}^2 + \ell^2\sin^2\theta\,\dot{\theta}^2\right) + mg\ell\cos\theta. \tag{2.13}$$

The derivatives required for Lagrange's equation are:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x}, \\ \frac{\partial\mathcal{L}}{\partial x} &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= m\ell^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^2\theta\,\dot{\theta}\right) \\ &= m\ell^2\left(2\sin\theta\cos\theta\,\dot{\theta}^2 + \sin^2\theta\,\ddot{\theta}\right) \\ \frac{\partial\mathcal{L}}{\partial\theta} &= m\ell^2\sin\theta\cos\theta\,\dot{\theta}^2 - mg\ell\sin\theta \end{aligned}$$

so Lagrange's equations give us:

$$\begin{aligned} m\ddot{x} &= 0 \\ m\ell^2\left(2\sin\theta\cos\theta\,\dot{\theta}^2 + \sin^2\theta\,\ddot{\theta}\right) &= m\ell^2\sin\theta\cos\theta\,\dot{\theta}^2 - mg\ell\sin\theta \end{aligned}$$

or, simplifying and rearranging, we get

$$\begin{cases} \ddot{x} &= 0 \\ \sin^2\theta\,\ddot{\theta} + \sin\theta\cos\theta\,\dot{\theta}^2 &= -\frac{g}{\ell}\sin\theta \end{cases}$$

You will notice the first equation is conservation of the x-component of momentum; physically, this is to be expected because both the constraining force and gravity act vertically so have no component in the x-direction. The second equation is singular when $\theta = 0$ or π, as the coefficient of $\ddot{\theta}$ vanishes there. This is closely related to the fact that the matrix $\mathbb{K}$ is not invertible at those points.

We will not attempt to say anything about the solutions here.

The general formula for $\mathbb{K}$ is found by substituting $\mathbf{v} = D\psi(q)\dot{\mathbf{q}}$ into the expression for the kinetic energy. The expression in Euclidean (Cartesian) coordinates can be written

as

$$T = \tfrac{1}{2}\mathbf{v}^T M\mathbf{v}$$

where M is the "mass matrix". If there is just one particle, then $M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$. In general, M is the diagonal matrix with diagonal entries $(m_1, m_1, m_1, m_2, m_2, m_2, \ldots, m_N, m_N, m_N)$; that is, in 3×3 blocks,

$$M = \begin{bmatrix} m_1 I_3 & 0 & 0 & 0 \\ 0 & m_2 I_3 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & m_N I_3 \end{bmatrix},$$

where I_3 is the 3×3 identity matrix. Thus substituting $\mathbf{v} = D\psi_q\dot{\mathbf{q}}$ gives

$$\begin{aligned} T = \tfrac{1}{2}\mathbf{v}^T M\mathbf{v} &= \tfrac{1}{2}(D\psi_q\dot{\mathbf{q}})^T M (D\psi_q\dot{\mathbf{q}}) \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^T (D\psi_q)^T M (D\psi_q\dot{\mathbf{q}}) \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^T \left(D\psi_q^T M D\psi_q\right)\dot{\mathbf{q}} \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^T \mathbb{K}(q)\dot{\mathbf{q}} \end{aligned}$$

so that

$$\mathbb{K}(q) = D\psi_q^T M D\psi_q. \tag{2.14}$$

If the system has n degrees of freedom, then $\mathbb{K}(q)$ is an $n \times n$ symmetric matrix.

REMARK 2.16. Notice that the kinetic energy T was originally just a function of the velocity $\mathbf{v} = \dot{\mathbf{x}}$, but is now a function of both position (configuration) and velocity; this is because when using non-Euclidean coordinates the Jacobian matrix $D\psi_q$ often brings q explicitly into the expression (2.14) for the kinetic energy. From now on, we will therefore always write $T(q, \dot{\mathbf{q}})$. ❞

DEFINITION 2.17. A Lagrangian $\mathcal{L} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbb{K}\,\dot{\mathbf{q}} - V(q)$ is ***regular*** or ***non-degenerate*** if $\mathbb{K}(q)$ is an invertible matrix ✔

Since one assumes all the masses of the particles in a system are non-zero, the mass matrix M is an invertible matrix. It then follows from (2.14) that $\mathbb{K}$ is degenerate iff $D\psi_q$ fails to be injective. In other words, the cause[5] of a Lagrangian to be degenerate is that the coordinate system is not regular, such as occurs with polar coordinates when $r = 0$ and $\det D\psi_{(0,\theta)} = 0$ (see equation (2.10)). In many cases, this can be resolved by choosing a better system of coordinates, but sometime the degeneracy is inherent to the system (as in the elliptic pendulum above).

The following definition extends the observation made in Remark 2.5.

DEFINITION 2.18. Consider a Lagrangian system with (generalized) coordinates $(q_1, \ldots, q_n)$.

[5] One can also imagine a system where one of the particles has zero mass in which case M is itself degenerate, and then $\mathbb{K}$ may also be degenerate even with regular coordinates.

For each coordinate q_j, the quantity

$$p_j := \frac{\partial \mathcal{L}}{\partial \dot{q}_j},$$

is called the ***generalized momentum*** corresponding to q_j, or *conjugate* to q_j. The quantity $\partial \mathcal{L}/\partial q_j$ is called the ***generalized force*** on the system, associated to q_j (or, conjugate to q_j). ✔

This quantity is named momentum because, in standard Euclidean coordinates for a particle, $\mathcal{L} = \frac{1}{2} m\dot{q}^2 - V(q)$, so that

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$$

which is precisely the usual momentum. And in Example 2.19 below, $\partial \mathcal{L}/\partial \dot{\theta}$ is the angular momentum about the z-axis, so $\partial \mathcal{L}/\partial \dot{q}$ encompasses both types of momentum we know.

For example, in the plane pendulum, $p = p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$, and this is equal to the angular momentum about the origin. In the elliptic pendulum above, we have from (2.13).

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \qquad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell^2 \sin^2\theta\, \dot{\theta}.$$

If, as in the definition, the quantity $\partial \mathcal{L}/\partial q$ is the generalized force, then Lagrange's equation reads,

generalized force equals rate of change of generalized momentum

which should sound familiar if you remove the word 'generalised'!

For general coordinates, if the system is a simple mechanical one with $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbb{K}(\mathbf{q})\dot{\mathbf{q}} - V(\mathbf{q})$, then

$$\mathbf{p} = \mathbb{K}(\mathbf{q})\dot{\mathbf{q}}.$$

In particular, the Lagrangian being regular is equivalent to the linear map $\dot{\mathbf{q}} \mapsto \mathbf{p} = \mathbb{K}\dot{\mathbf{q}}$ being *invertible*.

Example 2.19. (The spherical pendulum) This is a pendulum that is not confined to a plane, where the bob moves on a circle, but now the bob can move on a sphere. For coordinates we use spherical polar coordinates $q = (\theta, \varphi)$, with $\theta \in [0, \pi]$ measuring the angle with the downward vertical, and $\varphi \in [0, 2\pi)$ the angle between the arm of the pendulum and a fixed vertical plane (so $\theta = \pi/2 -$ latitude, and φ is the longitude). The potential energy is given by $V = mg\ell(1 - \cos\theta)$ (taking the downward configuration as zero). To find the kinetic energy, one needs to express the velocity in Euclidean terms: we have

$$\mathbf{r} = \psi(\theta, \varphi) = (\ell \sin\theta \cos\varphi,\ \ell \sin\theta \sin\varphi,\ -\ell \cos\theta)$$

and so

$$\dot{\mathbf{r}} = \ell\dot{\theta}(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) + \ell\dot{\varphi}(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0).$$

One then finds that the kinetic energy, which in Euclidean terms is $T = \frac{1}{2}m|\dot{\mathbf{r}}|^2$, becomes in spherical polar coordinates,

$$T = \tfrac{1}{2}m\ell^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right), \tag{2.15}$$

as the reader should check. The mass-inertia matrix is then given by

$$\mathbb{K}(\theta,\varphi) = \begin{pmatrix} m\ell^2 & 0 \\ 0 & m\ell^2\sin^2\theta \end{pmatrix}.$$

The Lagrangian fails to be regular when $\sin\theta = 0$, so at $\theta = 0, \pi$ which correspond to the pendulum being either upward or downward. This is because spherical polar coordinates are not regular at these points.

The Lagrangian is thus

$$\mathcal{L}(\mathbf{q},\dot{\mathbf{q}}) = \tfrac{1}{2}m\ell^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - mg\ell(1-\cos\theta).$$

Lagrange's equations for this system are then, after simplifying,

$$\begin{cases} \ddot{\theta} &= \sin\theta\cos\theta\,\dot{\varphi}^2 - \frac{g}{\ell}\sin\theta \\ \frac{d}{dt}(m\ell^2\sin^2\theta\,\dot{\varphi}) &= 0. \end{cases}$$

The final 0 arises because $\mathcal{L}$ is independent of φ so $\partial\mathcal{L}/\partial\varphi = 0$. This corresponds to there being no force in the φ-direction. In general, if $\mathcal{L}$ is independent of a particular coordinate, that coordinate is said to be ***cyclic***, so in this example the coordinate φ is cyclic. The quantity $\partial\mathcal{L}/\partial\dot{\varphi} = m\ell^2\sin^2\theta\,\dot{\varphi}$ is the angular momentum of the pendulum around the vertical axis.

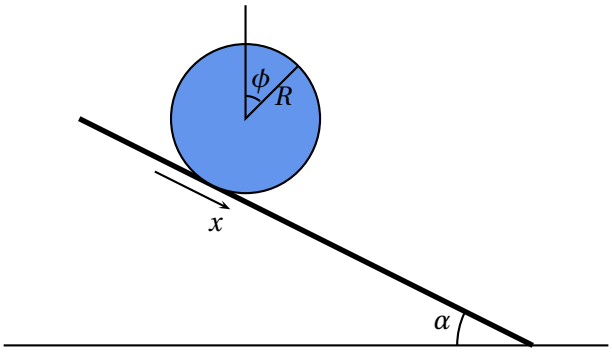


FIGURE 2.3: Cross-section of a cylinder rolling down an inclined plane

EXAMPLE 2.20. (Cylinder rolling down inclined plane)

Consider a uniform cylinder of mass M, radius R and length ℓ, rolling without slipping down a plane inclined at an angle α to the horizontal. Find the equation of motion.

Solution: Referring to Figure 2.3 we can choose either x or ϕ as coordinate. Here x is the distance moved along the plane (eg by the centre of mass), and ϕ is the angle made by a fixed radial line with the vertical. Provided one starts with $x = \phi = 0$, the two are related by the rolling condition

$$x = R\phi.$$

To find the Lagrangian, we need first to calculate the kinetic energy. Recall from Chapter 1, the kinetic energy is the sum of the kinetic energy "of the centre of mass", whose velocity here is $\dot{x}$, and the kinetic energy T_0 of the motion of the particles relative to the centre of mass:

$$T = \tfrac{1}{2}M\dot{x}^2 + T_0.$$

Now T_0 is the energy due to the rotation about the axis of the cylinder, which as we saw in Chapter 1 is

$$T_0 = \tfrac{1}{2}I\omega^2,$$

where $I = \frac{1}{2}MR^2$, the moment of inertia about the axis of the cylinder. Thus, using ϕ as the coordinate, we have $\omega = \dot{\phi}$ and

$$T = \tfrac{1}{2}M\dot{x}^2 + T_0 = \tfrac{1}{2}MR^2\dot{\phi}^2 + \tfrac{1}{2}I\dot{\phi}^2 = \tfrac{3}{2}I\dot{\phi}^2.$$

The potential energy is gravitational, so $V = Mgh = -Mgx\sin\alpha = -MgR\phi\sin\alpha$ (up to adding a constant), so that the Lagrangian is given by

$$\mathcal{L}(\phi,\dot{\phi}) = \tfrac{3}{2}I\dot{\phi}^2 + MgR\phi\sin\alpha.$$

Lagrange's equations now read

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(3I\dot{\phi}\right) = MgR\sin\alpha.$$

so that

$$3I\ddot{\phi} = MgR\sin\alpha.$$

Substituting for $I = \frac{1}{2}MR^2$, the acceleration of the cylinder is therefore

$$\ddot{\phi} = \frac{2g\sin\alpha}{3R}.$$

Or, with $x = R\phi$, $\ddot{x} = \frac{2}{3}g\sin\alpha$.

REMARK 2.21. In that last example, one can consider x, ϕ as independent variables, so

describing a system with 2 degrees of freedom. This would in fact be necessary if the cylinder were allowed to slide down the slope. However, the no-slip (or rolling) condition is a holonomic constraint imposing the relationship $x = R\phi$ between the two variables, thus reducing it to the 1 degree of freedom system we considered above. ❞

All our Lagrangians have been of the form "kinetic minus potential", but others do arise. For example, a charged particle in an electromagnetic field is subject to the Lorentz force (see §1.6) $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The Lagrangian for this is

$$\mathcal{L}(\mathbf{r}, \mathbf{v}) = \tfrac{1}{2} m \|\mathbf{v}\|^2 - q\phi(\mathbf{r}) + q\mathbf{A}(\mathbf{r}) \cdot \mathbf{v}, \tag{2.16}$$

where $\mathbf{r}$ is the position vector of the particle, $\mathbf{v} = \dot{\mathbf{r}}$ its velocity, $\phi(\mathbf{r})$ is the electric potential, so $\mathbf{E} = \vec{\nabla}\phi$, and $\mathbf{A}$ is the magnetic vector potential: $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$. Notice that the last term is linear in the velocity, so is neither kinetic nor potential energy.

EXERCISE Show Lagrange's equations for (2.16) do indeed give the same equation of motion as Newton's laws with the Lorentz force (Problem 2.12). ☆

REMARK 2.22. If there is in addition a non-conservative force $\mathbf{F}$ acting on the system, then one can modify Lagrange's equation to take this into account. If we write $\mathbf{F}_i$ as this non-conservative force acting on particle i, and write $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_K)$ then the result is

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_j}\right) = \frac{\partial \mathcal{L}}{\partial q_j} + Q_j$$

where $Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$, so as a vector, $\mathbf{Q} = \mathbf{F}^T D\psi_q$. The quantity $\partial\mathcal{L}/\partial q_j + Q_j$ is called a ***generalized force*** corresponding to the generalized coordinate q_j. ❞

2.5 NOETHER'S THEOREM

Notice that in Example 2.19 above, the fact that ϕ is a cyclic coordinate leads to a conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(m\ell^2 \sin^2\theta\,\dot{\phi}\right) = 0,$$

or in other words, $m\ell^2 \sin^2\theta\,\dot{\phi}$ is constant. This is a particular case of a general phenomenon that we will be returning to and using many times:

THEOREM 2.23 (Noether). *Let $\mathcal{L}(\mathbf{q}, \mathbf{v})$ be a Lagrangian for a given system, and suppose that the coordinate q_1 is cyclic (i.e. $\mathcal{L}$ is independent of q_1). Then the quantity $\partial\mathcal{L}/\partial\dot{q}_1$ is conserved.*

PROOF: This follows immediately from Lagrange's equations (as in the example above), for

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) = \frac{\partial \mathcal{L}}{\partial q_1},$$

and this latter term is 0 as $\mathcal{L}$ is independent of q_1 ❐

Since $p = \partial\mathcal{L}/\partial\dot{q}$ is the momentum associated to q, Noether's theorem tells us that if a variable q is cyclic then the corresponding momentum is conserved.

This result of Emmy Noether[6] is of great importance as it helps restrict the types of motion possible in a given system. In the spherical pendulum example this result was easy to apply as it 'jumped out' from the equations of motion. However, this is not always the case, and one may have to make a judicious change of coordinates to see a cyclic variable, as in the following example.

Example 2.24. Recall Example 2.7 where we considered two particles of m_1 and m_2 attracted by their gravitational interaction (see also Exercise 1.6). The Euclidean coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ are not cyclic, but a change of coordinates brings out the law of conservation of momentum. Write $M = m_1 + m_2$ and $\mu_j = m_j/M$ $(j = 1,2)$, so that $\mu_1 + \mu_2 = 1$. Define $\mathbf{q} = (\mathbf{q}_+, \mathbf{q}_-)$ by

$$\mathbf{q}_+ = \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2, \qquad \mathbf{q}_- = \mathbf{r}_1 - \mathbf{r}_2.$$

That is, $\mathbf{q}_+$ is the position of the centre of mass and $\mathbf{q}_-$ the relative position of the particles. This change of coordinates can easily be inverted:

$$\mathbf{r}_1 = \mathbf{q}_+ + \mu_2\mathbf{q}_-, \quad \mathbf{r}_2 = \mathbf{q}_+ - \mu_1\mathbf{q}_-.$$

Write this as $(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{q}_+, \mathbf{q}_-)$. We want to express the Lagrangian in terms of these new coordinates, and for that we use the differential of ψ and replace the Lagrangian (2.6) by $\mathcal{L}' = \mathcal{L} \circ D\psi$:

$$\mathcal{L}' = \tfrac{1}{2}M\|\mathbf{v}_+\|^2 + \frac{m_1m_2}{2M}\|\mathbf{v}_-\|^2 + \frac{Gm_1m_2}{\|\mathbf{q}_-\|},$$

where $\mathbf{v}_+ = \dot{\mathbf{q}}_+$ etc. Now all three components of $\mathbf{q}_+$ are cyclic as $\mathbf{q}_+$ does not appear in the expression for $\mathcal{L}'$. Noether's theorem therefore implies that the three components of

$$\frac{\partial\mathcal{L}'}{\partial\mathbf{v}_+} = M\mathbf{v}_+$$

are conserved. But $M\mathbf{v}_+$ is the total momentum $\mathbf{p}$ as defined in §1.3, so we have shown that the total momentum is conserved (a fact we already proved in Corollary 1.3 on p. 8).

Finally, notice that in this example and in these coordinates, the mass-inertia matrix (see Definition 2.17) is

$$\mathbb{K} = \begin{pmatrix} M & 0 \\ 0 & \frac{m_1m_2}{M} \end{pmatrix},$$

[6] Emmy Noether: 1882–1935. Born in Germany, she emigrated to the US to avoid Nazi persecution. She made many contributions to mathematics, mostly in algebra. She proved a result that implies the theorem above in 1915, the same year Einstein's paper on General Relativity appeared. Einstein is said to have been impressed by the power and scope of Noether's theorem which goes far beyond the statement above.

which is clearly positive definite, so the Lagrangian is regular. (Recall that a symmetric matrix S (or quadratic form) is positive definite if for all $\mathbf{x} \neq 0$, one has $\mathbf{x}^T S\mathbf{x} > 0$, or equivalently if all the eigenvalues of S are positive.)

2.6 * ROUTH REDUCTION

Edward Routh in his famous prize-winning essay in 1877 developed (among other things) a systematic method for taking advantage of the conserved quantities arising in Noether's theorem; this is now called Routh reduction. We begin with an example where there is just one cyclic coordinate: the planar central force problem.

EXAMPLE 2.25. Consider a particle of mass m in the plane attracted by a gravitational force to the origin. We will use polar coordinates to take advantage of the cyclic variable. The potential energy is $V = -k/r$ for some constant k, and the kinetic energy is $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. The Lagrangian is

$$\mathcal{L}(r,\theta,\dot{r},\dot{\theta}) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}.$$

Clearly (and as we have seen before) θ is cyclic, and $p_\theta = mr^2\dot{\theta}$ is conserved (it is the angular momentum about the origin — see Chapter 1). Fix a value μ for p_θ ('μ' for momentum) and define the *modified Lagrangian* or ***Routhian*** for the value μ by

$$\mathcal{R}_\mu(r,\dot{r}) = \mathcal{L}(r,\dot{\theta},\dot{r}) - \mu\dot{\theta}$$

and substituting for $\dot{\theta}$ in terms of $\mu = p_\theta$ to get

$$\begin{aligned}
\mathcal{R}_\mu(r,\dot{r}) &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} - \mu\dot{\theta} \\
&= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\left(\frac{\mu}{mr^2}\right)^2 + \frac{k}{r} - \mu\left(\frac{\mu}{mr^2}\right) \\
&= \tfrac{1}{2}m\dot{r}^2 + \frac{k}{r} - \frac{\mu^2}{2mr^2}.
\end{aligned}$$

Notice that this is the Lagrangian of a 1-degree of freedom system, and there is one for each value of μ. Moreover, we can interpret the first term as kinetic energy and the second two together as minus potential energy:

$$V_\mu(r) = -\frac{k}{r} + \frac{\mu^2}{2mr^2}.$$

This is equal to the original potential energy plus an extra term arising from the momentum value μ (and in this example can be interpreted as a centrifugal contribution). This is called the ***amended potential*** for the reduced problem.

The general procedure introduced by Routh is as follows.

Consider a Lagrangian system given by $\mathcal{L}(\mathbf{q},\dot{\mathbf{q}})$, where some of the coordinates are cyclic. Let us write $q = (\theta_1,\ldots,\theta_k,x_1,\ldots,x_r)$, where the θ_i are all cyclic, while the x_i are not; so the total number of degrees of freedom is $n = k+r$. Then by Noether's theorem the momenta $p_{\theta_1},\ldots,p_{\theta_k}$ are conserved quantities.

The aim is to treat the p_{θ_j} as constants (so parameters in some sense) and the remaining variables $x_1,\ldots,x_r$ (and the corresponding velocities) as the variables in a smaller 'reduced' Lagrangian system.

The reduction proceeds by eliminating from $\mathcal{L}$ the $\dot{\theta}$ variables using

$$p_{\theta_j} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_j} \quad (j = 1,\ldots,k). \tag{2.17}$$

So we need to assume that one can solve (2.17) for each $\dot{\theta}_j$ as a function of the p_{θ_i}; this is always possible if $\mathcal{L} = T - V$ with $T = \frac{1}{2}\dot{\mathbf{q}}^T\mathbb{K}\dot{\mathbf{q}}$ and $\mathbb{K}$ is positive definite. Define the ***Routhian*** $\mathcal{R}_\mu(x,\dot{x})$ by[7]

$$\mathcal{R}_\mu(\mathbf{x},\dot{\mathbf{x}}) = \mathcal{L}(\theta,\mathbf{x},\dot{\theta},\dot{\mathbf{x}}) - \sum_{j=1}^{k}\mu_j\dot{\theta}_j,$$

after eliminating the $\dot{\theta}_j$ in favour of the $\mu_j = p_{\theta_j}$ using (2.17), as was done in the example above.

THEOREM 2.26 (Routh, 1877). *Consider the Lagrangian $\mathcal{L}(\theta,\mathbf{x},\dot{\theta},\dot{\mathbf{x}})$ and suppose the θ coordinates are cyclic and that $\mathcal{L}$ is R-regular. Then when the conserved quantity $p_\theta = \mu$ the variables $\mathbf{x}$ evolve as if governed by the Routhian $\mathcal{R}_\mu$ defined above.*

PROOF: Now $\mathbf{x}$ evolves according to Lagrange's equations for $\mathcal{L}$. Since $\dot{\theta}$ depends on $\mathbf{x}$ and $\dot{\mathbf{x}}$ (and the constant μ) we have

$$\begin{aligned}
\frac{\partial \mathcal{R}_\mu}{\partial x_i} &= \frac{\partial \mathcal{L}}{\partial x_i} + \sum_j \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_j}\right)\left(\frac{\partial \dot{\theta}_j}{\partial x_i}\right) - \sum_j \mu_j \frac{\partial \dot{\theta}_j}{\partial x_i} \\
&= \frac{\partial \mathcal{L}}{\partial x_i} + \sum_j \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_j} - \mu_j\right)\left(\frac{\partial \dot{\theta}_j}{\partial x_i}\right) \\
&= \frac{\partial \mathcal{L}}{\partial x_i}
\end{aligned}$$

where the partial derivatives are taken at constant $p_\theta = \mu$. And likewise (with $\dot{x}_i$ replacing x_i),

$$\frac{\partial \mathcal{R}_\mu}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}.$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{R}_\mu}{\partial \dot{x}_i}\right)$ and therefore because $\mathbf{x}$ satisfies Lagrange's equations for $\mathcal{L}$, it also satisfies Lagrange's equations for $\mathcal{R}_\mu$, as required. ❐

[7] Routh called this the *modified Lagrangian*

One thing to note is that even if the original system describes a simple mechanical system the reduced one may not be of the same form. However, it can always be written

$$\mathcal{R}_\mu(x,\dot{x}) = \tfrac{1}{2}\dot{x}^T\mathbb{K}\dot{x} + A(\mu)\dot{x} - V_\mu(x).$$

Here $A(\mu)$ depends linearly on μ (and depends on the original form of the kinetic energy). The quantity $V_\mu(x)$, which involves all the terms independent of $\dot{x}$ is called the ***amended potential***. In the example above, $A(\mu) = 0$ and $V_\mu(r) = V(r) + \mu^2/2mr^2$.

DEFINITION 2.27. A motion where all the x-coordinates are constant and only the cyclic coordinates vary is called a ***relative equilibrium*** or *steady motion*. ✔

In this case one can show that the cyclic variables always vary uniformly; that is, $\dot{\theta}$ is constant. Thus a relative equilibrium is an equilibrium point for the reduced system governed by the Routhian.

Routh's motivation in his work was to find a method for determining whether a relative equilibrium is stable or not, and his 'reduction' procedure reduces the problem to that of the stability of an equilibrium point, using the amended potential. See the next chapter for how stability is determined from the potential energy.

2.7 * HAMILTON'S PRINCIPLE

This section is included for its own intrinsic interest, and not used elsewhere in the book, so can be safely skipped. The principal motivation for including it is that it provides a beautiful argument justifying Lagrange's equations, so providing a proof of Theorem 2.13.

William Rowan Hamilton[8] was a very important mathematician and contributed to a number of areas. His first major contribution to science was in the field of optics, when he published an important paper at the age of 21. In his work on optics he was particularly fascinated by Fermat's principle of least time, which states that for a light ray to go from a point A to a point B through any medium, it will choose the path of shortest time. One can easily deduce the laws of reflection and refraction from this principle (see eg Feynman's Lectures in Physics, Volume I).

When Hamilton turned his mind to mechanics he was probably struck by the fact that like light, particles travel in straight lines unless deflected by some external force (analogous to deflecting a light ray by reflection or refraction), so he asked whether there could be a principle similar to Fermat's that governs mechanics. Of course, it could not simply be a least time principle as a particle can travel from A to B in various times according to how fast it travels.

Hamilton's solution to his problem is as follows. Let $\mathcal{L}(q,\mathbf{v})$ be the Lagrangian for the mechanical system. Fix the points A and B in the configuration space, and *fix* a time interval T. Consider a smooth path $\gamma(t)$ in the configuration space with $\gamma(0) = A$ and

[8] Sir William Rowan Hamilton, 1805–1865, Astronomer Royal of Ireland

$\gamma(T) = B$, and to such a path associate a real number, the ***action***,

$$S(\gamma) = \int_0^T \mathcal{L}(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}\,t.$$

This integral depends on the path chosen.

Hamilton's principle, also known as the principle of least action or more correctly, of stationary action, states that the path taken by the mechanical system to go from A to B in time T is precisely the one for which the action is stationary (that is, it is a critical point of the action, although not usually a minimum). This means that, if we write

$$\gamma_\lambda(t) := \gamma(t) + \lambda\eta(t)$$

with $\eta(0) = \eta(T) = 0$, which for λ small is a nearby path satisfying the same conditions $\gamma_\lambda(0) = A$ and $\gamma_\lambda(T) = B$, then we require

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S(\gamma_\lambda)\Big|_{\lambda=0} = 0,$$

and we require this for *every* possible perturbation $\eta(t)$.

THEOREM 2.28 (Hamilton)**.** *Hamilton's principle is equivalent to Lagrange's equations.*

PROOF: The action of the path γ_λ is:

$$S(\gamma_\lambda) = \int_0^T \mathcal{L}(\gamma(t) + \lambda\eta(t), \dot{\gamma}(t) + \lambda\dot{\eta}(t))\,\mathrm{d}\,t.$$

Differentiating with respect to λ at $\lambda = 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S(\gamma_\lambda)\Big|_{\lambda=0} = \int_0^T \left(\frac{\partial\mathcal{L}}{\partial q}\eta(t) + \frac{\partial\mathcal{L}}{\partial\mathbf{v}}\dot{\eta}(t)\right)\mathrm{d}\,t.$$

Integrating the second term by parts gives

$$\int_0^T \frac{\partial\mathcal{L}}{\partial\mathbf{v}}\dot{\eta}(t)\,\mathrm{d}\,t = \left[\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\eta(v)\right]_0^T - \int_0^T \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right)\eta(t)\,\mathrm{d}\,t.$$

Since $\eta(0) = \eta(T) = 0$ the first term on the right hand side vanishes, so that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S(\gamma_\lambda)\Big|_{\lambda=0} = \int_0^T \left[\frac{\partial\mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right)\right]\eta(t)\,\mathrm{d}\,t.$$

The terms $\partial\mathcal{L}/\partial q$ and $\partial\mathcal{L}/\partial\mathbf{v}$ are of course evaluated at $(\gamma(t), \dot{\gamma}(t))$. Note that the term in square brackets is just the left hand side of Lagrange's equations.

It is therefore clear that if $\gamma(t)$ is a path that satisfies Lagrange's equations, then for any η, $\mathrm{d}S(\gamma_\lambda)/\mathrm{d}\lambda = 0$.

Conversely, if

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S(\gamma_\lambda)\Big|_{\lambda=0} = 0$$

for all perturbations, so for all η, then $\gamma(t)$ satisfies Lagrange's equation. This is because:

LEMMA 2.29. *If a continuous function $f : [0, T] \to \mathbb{R}$ satisfies*

$$\int_0^T f(t)\eta(t)\,\mathrm{d}\,t = 0$$

for all smooth functions η satisfying $\eta(0) = \eta(T) = 0$ then $f = 0$.

This is not hard to prove. For details see any book on the calculus of variations. ❐

LAGRANGE'S EQUATIONS AND NEWTON'S LAWS An easy consequence of Hamilton's principle is the important Theorem 2.13. Indeed, since the action associated to a parametrized path does not depend on which particular coordinate system is being used (it depends only on what units of time one uses, since the integral is over time), it follows that Hamilton's principle does not depend on the choice of coordinate system. Consequently, Lagrange's equations are invariant. And we have already shown that these equations coincide with Newton's equations in Cartesian (Euclidean) coordinates.

There is another important principle in mechanics, called Maupertuis principle, which is closely related to, and predates, Hamilton's principle and can be found in many books.

REMARK 2.30. The argument we have been using assumes that the Lagrangian is independent of time. If not, then one needs to specify not only the length T of the time interval, but also the precise interval over which the motion takes place: $[t_1, t_2]$ and then the action is defined in the same manner:

$$S(\gamma) = \int_{t_1}^{t_2} \mathcal{L}(\gamma(t), \dot\gamma(t), t)\,\mathrm{d}\,t. \qquad \text{❞}$$

PROBLEMS

2.1 Let $V(x, y) = -3xy$ be the potential energy function governing the motion of a particle of mass m moving in the plane. Determine the Lagrangian for this system and deduce the equations of motion. (You are not expected to try to solve them.)

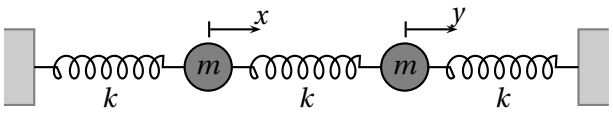
2.2 Show that the equations of motion in Equations (2.5) (Cartesian) and (2.7) (polar) are equivalent.
[Hint: Differentiate $x = r\cos\theta$ and $y = r\sin\theta$ twice each. The first time should give you $\dot x = \dot r\cos\theta - r\dot\theta\sin\theta$, and something similar for $\dot y$. Then $\ddot x = \ddot r\cos\theta - 2\dot r\dot\theta\sin\theta - r\ddot\theta\sin\theta - r\dot\theta^2\cos\theta$ and something similar for $\ddot y$. By considering $\ddot x\cos\theta + \ddot y\sin\theta$ and $\ddot x\sin\theta - \ddot y\cos\theta$ eliminate x and y from the equations (2.5).]

2.3 (i) Let $V(x, y) = \frac{1}{4}(x^2 + y^2)^2$ be the potential energy function governing the motion of a particle of mass m moving in the plane. Determine the Lagrangian for this system and deduce the equations of motion.

(ii) Let $V(r,\theta) = \frac{1}{4}r^4$ be the potential energy function governing the motion of a particle of mass m moving in the plane. Using polar coordinates as generalized coordinates, determine the Lagrangian for this system, and deduce the equations of motion. [Partial answer: One of the equations is $m\ddot r = mr\dot\theta^2 - r^3$.]

2.4 Consider a system of two identical interacting particles of mass m with potential $V(r)$, where r is the inter-particle distance. Let $\mathbf{r}_G$ be the position vector of the centre of mass, and let $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ be the position vector of particle 1 relative to particle 2. Find the Lagrangian using as generalized coordinates $\mathbf{r}_G$ and $\mathbf{r}$.
[Hint: first write $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{r}_G$ and $\mathbf{r}$.]

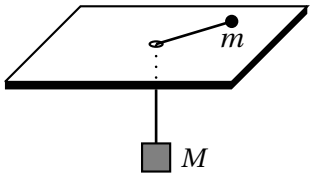
2.5 Consider the system consisting of two identical masses of mass m that can move horizontally, joined with three identical springs with spring constant k as shown below. Denote by x, y the horizontal displacements of the two masses.



Write down the Lagrangian for this system and deduce its equations of motion. [We will solve these equations in the next chapter.] [Partial answer: for the y coordinate, $m\ddot{y} = kx - 2ky$.]

2.6 Determine the Lagrangian and deduce the equations of motion for the following system:

A particle of mass m moves on a smooth table, and is attached to a light inextensible string of length ℓ passing through a smooth hole in the table, the other end of which is attached to a weight of mass M. [*Suggestion*: for generalized coordinates, use the polar coordinates of the position of the particle on the table, with centre at the hole: why is this enough?]



Find the relation between the height of M and the angular velocity of m required in order to maintain a motion in which M is stationary, and m rotates uniformly. (Such a motion is often called a *relative equilibrium*.) [Partial answer: the relation asked for at the end is $\omega^2 = Mg/m(\ell - z)$, where z is the distance of M below the table]

2.7 Find the Lagrangian for the *Atwood machine* from Ex. 1.7 and deduce the equations of motion. Do this both for a light pulley, and for a uniform pulley with moment of inertia I_0.

2.8 A *spring pendulum* is a pendulum where the rod is no longer inextensible, but is modelled as a light spring. Determine the Lagrangian of such a system (you will need to give symbols to represent the physical quantities involved), and derive its equations of motion.

2.9 Let $\mathcal{L}(q, \mathbf{v})$ with $q = (q_1, \ldots, q_n)$, be the Lagrangian for a given system. Define the ***energy function*** E by

$$E(q, \mathbf{v}) = \sum_i v_i \frac{\partial \mathcal{L}}{\partial v_i} - \mathcal{L}(q, \mathbf{v}).$$

Show using Lagrange's equation, that the energy function is a conserved quantity.

2.10 Identify a cyclic variable in problem 2.6 and apply Noether's theorem to the system to find the corresponding conserved quantity. Now apply the procedure of Routh reduction to deduce a 1-degree of freedom problem for the non-cyclic variable. Find the amended potential and the value of r giving a relative equilibrium with momentum μ. Determine whether it is stable or unstable.

2.11 Non-uniqueness of the Lagrangian: let $\mathcal{L}' = a\mathcal{L} + b$ with a, b constants and $a \neq 0$. Show that Lagrange's equations arising from $\mathcal{L}$ and from $\mathcal{L}'$ are identical.

2.12 (i) Show that the Lagrangian for a charged particle in an electromagnetic field given in (2.16) gives the same equations of motion as Newton's law with the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. [This is a lengthy exercise in vector calculus.]

(ii) Show that for the same Lagrangian, the energy E as defined in problem 2.9 is independent of the magnetic field.

2.13 Consider a charged particle moving in the uniform magnetic field $\mathbf{B} = \mathbf{e}_3$ (the unit vector in the z-direction), with no electric field. Show that $\mathbf{A} = (0, x, 0)^T$ is a vector potential for $\mathbf{B}$ and find the Lagrangian for the particle. Write down the generalized momenta for each of the coordinates. Use Noether's theorem to show that the z-component of the usual[9] momentum is conserved.

2.14 Consider the following *compound Atwood machine*. A smooth light pulley is attached to a fixed support, and a light inextensible string is passed over it. To one end of the string is attached a weight of mass $m_1 = 8\,\text{kg}$ and to the other a smooth light pulley. Over this second pulley is passed another light inextensible string with a weight attached at each end, of masses $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$ respectively. The masses are subject to gravity. How many degrees of freedom does the system have? Find the Lagrangian of the system, and hence the accelerations of the three masses. Deduce that if they start from rest, then one of the masses does not move.
[Hint: Let x, y, z be vertical displacements of the three weights respectively. Show that $2x + y + z$ is constant.]

[9] as against generalized momentum